Then we have

$$\begin{aligned} s_1 &= \lambda_1\lambda_2 + \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2\mu_1, \\ s_2 &= \lambda_1\lambda_2 - \mu_1\mu_2 - \lambda_1\mu_2 + \lambda_2\mu_1, \\ t_1 &= \lambda_1\lambda_2 - \mu_1\mu_2 + \lambda_1\mu_2 - \lambda_2\mu_1, \\ t_2 &= \lambda_1\lambda_2 + \mu_1\mu_2 + \lambda_1\mu_2 + \lambda_2\mu_1. \end{aligned} \tag{12}$$

It is now clear that equations (8), (10) and (12) afford a four-parameter solution of (6).

In the special case when $\mu_1 = 0$ we have $s_1 = s_2$, $t_1 = t_2$. If we write $s_1 = s_2 = \alpha$, $t_1 = t_2 = \beta$, then the solution afforded by (8), (10), (12) becomes identical with solution (7). Hence (8), (10), (12) afford a solution which is a generalization of the previously known solution (7).

As illustrative examples of the solution of (6) afforded by (8), (10), (12), we have the following:

$b = 3$, $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 2$: $142^2 + 3\cdot 78^2 = 14^4$,

$b = 5$, $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 2$: $439^2 + 5\cdot 132^2 = 23^4$,

$b = 2$, $\lambda_1 = 3$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 3$: $17^2 + 2\cdot 56^2 = 9^4$.

It should be noticed that the above solutions were not obtained merely by substitution. Factors common to u, v, x^2 have been removed after substitution in the formulæ.

§ 3. Case when b = 0. The equation $x^4 + ay^4 = u^2$. Consider the equation

$$x^4 + ay^4 = u^2. \tag{13}$$

Let $u = \alpha^2 + a\beta^2$. Then $u^2 = (\alpha^2 - a\beta^2)^2 + a(2\alpha\beta)^2$. Equation (13) will therefore be satisfied if we write

$$\begin{aligned} x^2 &= \alpha^2 - a\beta^2, \\ y^2 &= 2\alpha\beta, \\ u &= \alpha^2 + a\beta^2. \end{aligned} \tag{14}$$

The first of equations (14) is satisfied if

$$\begin{aligned} x &= \alpha_1^2 - a\beta_1^2, \\ \alpha &= \alpha_1^2 + a\beta_1^2, \\ \beta &= 2\alpha_1\beta_1. \end{aligned} \tag{15}$$

Now, putting $\alpha_1 = \lambda^2$, $\beta_1 = \mu^2$ in (15) and combining equations (14) and (15), we have

$$\begin{aligned} x &= \lambda^4 - a\mu^4, \\ y &= 2\lambda\mu\sqrt{\lambda^4 + a\mu^4}, \\ u &= \lambda^8 + 6a\lambda^4\mu^4 + a^2\mu^8. \end{aligned} \tag{16}$$

Hence, if λ and μ are values of x and y, respectively, satisfying (13), it is obvious that (16) will afford a second solution. Therefore, by a repeated use of (16), as many solutions as desired may be found.

Formulæ (16) were first obtained by Desboves (l. c., p. 437) by a method equivalent to that employed above.

From the second of equations (14) it follows that either α or β must contain 2 to an odd power. We shall first suppose it is α and write $\alpha = 2m\rho^2$, where m contains no repeated prime factor. Then, from the second equation (14) we see that we must have $\beta = m\sigma^2$. Now, substituting these values of α and β in the first equation (14), we see that values of ρ and σ satisfying the condition

$$4\rho^4 - a\sigma^4 = \tau^2 \tag{17}$$

will yield solutions of (13). If β contains 2 to an odd power, by reasoning similar to that employed above, we may write $\alpha = m\rho^2$, $\beta = 2m\sigma^2$. Making this substitution in the first equation (14), we see that values of ρ and σ satisfying the condition

$$\rho^4 - 4a\sigma^4 = \tau^2 \tag{18}$$

will yield solutions of (13).

Since equation (13) is known to have no solution when $a = 1$, we can not obtain a general solution free of restricting conditions. However, when $a = 2\lambda\mu(\lambda^2 + \mu^2)$, it can readily be shown that a solution is afforded by

$$\begin{aligned} x &= (\lambda - \mu)^2, \\ y &= 2(\lambda + \mu), \\ u &= \lambda^4 + 12\lambda^3\mu + 6\lambda^2\mu^2 + 12\lambda\mu^3 + \mu^4. \end{aligned} \tag{19}$$

When $a = 8\lambda\mu(\lambda^2 + \mu^2)$, it can readily be shown that a solution is afforded by

$$\begin{aligned} x &= (\lambda - \mu)^4 - 8\lambda\mu(\lambda^2 + \mu^2), \\ y &= 2(\lambda - \mu)(\lambda + \mu)^2, \\ u &= (\lambda - \mu)^8 + 48\lambda\mu(\lambda^2 + \mu^2)(\lambda - \mu)^4 + 64\lambda^2\mu^2(\lambda^2 + \mu^2)^2. \end{aligned} \tag{20}$$

Also, it is obvious that if a is a perfect square diminished by unity, integral solutions are possible. Solutions may be obtained when a has other special forms.

The following taken in order are examples illustrating the use of formulæ (16), (19), (20):

$$a = 3, \quad \lambda = 1, \quad \mu = 2: \quad 47^4 + 3 \cdot 28^4 = 2593^2,$$

$$a = 20, \quad \lambda = 2, \quad \mu = 1: \quad 1^4 + 20 \cdot 6^4 = 161^2,$$

$$a = 80, \quad \lambda = 2, \quad \mu = 1: \quad 79^4 + 80 \cdot 18^4 = 6881^2.$$

§ 4. Case when b = a. The equation $x^4 + ay^4 = u^2 + av^2$. From formulæ (3) we see that the equation

$$x^4 + ay^4 = u^2 + av^2 \tag{21}$$

is satisfied if

$$\begin{aligned} x^2 &= s_1s_2 - at_1t_2, \\ y^2 &= s_1t_2 + s_2t_1, \\ u &= s_1s_2 + at_1t_2, \\ v &= s_1t_2 - s_2t_1. \end{aligned} \tag{22}$$

Let $s_1 = \lambda - 3a\mu$, $s_2 = \lambda$, $t_1 = \lambda + a\mu$, $t_2 = -\mu$. Then equations (22) become

$$\begin{aligned} x &= \lambda - a\mu, \\ y^2 &= \lambda^2 + (a-1)\lambda\mu + 3a\mu^2, \\ u &= \lambda^2 - 4a\lambda\mu - a^2\mu^2, \\ v &= 3a\mu^2 - (a+1)\lambda\mu - \lambda^2. \end{aligned} \tag{23}$$

It may be easily verified (cf. R. D. Carmichael, l. c., p. 25) that

$$(m^2+amn+bn^2)^2 = (m^2-bn^2)^2+a(m^2-bn^2)(2mn+an^2)+b(2mn+an^2)^2. \tag{24}$$

Therefore, from the second of equations (23), we may write

$$\begin{aligned} y &= m^2 + (a-1)mn + 3an^2, \\ \lambda &= m^2 - 3an^2, \\ \mu &= 2mn + (a-1)n^2. \end{aligned} \tag{25}$$

It is now evident that equations (23) and (25) afford a two-parameter solution of (21).

Again, make the following substitution in (22): $s_1 = \lambda - 3a\mu$, $s_2 = \lambda + a\mu$, $t_1 = a\mu$, $t_2 = \lambda$. We then obtain

$$\begin{aligned} x^2 &= \lambda^2 - (a^2+2a)\lambda\mu - 3a^2\mu^2, \\ y &= \lambda - a\mu, \\ u &= \lambda^2 + (a^2-2a)\lambda\mu - 3a^2\mu^2, \\ v &= \lambda^2 - 4a\lambda\mu - a^2\mu^2. \end{aligned} \tag{26}$$

Making use of (24), we see that the first of equations (26) is satisfied if

$$\begin{aligned} x &= m^2 - (a^2+2a)mn - 3a^2n^2, \\ \lambda &= m^2 + 3a^2n^2, \\ \mu &= 2mn - (a^2+2a)n^2. \end{aligned} \tag{27}$$

It is now clear that equations (26) and (27) afford a two-parameter solution of (21).

It should be observed that the above methods are capable of yielding an unlimited number of two-parameter solutions. Thus solutions may be obtained by making the following substitutions in (22):

$$s_1 = \lambda + 2a\mu, \quad s_2 = \lambda, \quad t_1 = -2\mu, \quad t_2 = \lambda + 2a\mu,$$
$$s_1 = \lambda + 3a\mu, \quad s_2 = \lambda, \quad t_1 = -\mu, \quad t_2 = \lambda + 4a\mu,$$
$$s_1 = \lambda - 8a\mu, \quad s_2 = \lambda, \quad t_1 = -4\mu, \quad t_2 = \lambda + a\mu,$$
$$s_1 = \lambda + 3a\mu, \quad s_2 = \lambda, \quad t_1 = -3\mu, \quad t_2 = \lambda + 3a\mu.$$

From the above values of s_1, s_2, t_1, t_2 integers x are obtained by mere substitution. The corresponding expressions for y^2 contain λ^2 with coefficient unity. Similarly, the following substitutions yield y directly, the expressions for x^2 containing λ^2 with coefficient unity:

$$s_1 = \lambda, \quad s_2 = \lambda + a\mu, \quad t_1 = 4a\mu, \quad t_2 = \lambda,$$
$$s_1 = \lambda + 2a\mu, \quad s_2 = \lambda + 2a\mu, \quad t_1 = 2a\mu, \quad t_2 = \lambda,$$
$$s_1 = \lambda + 3a\mu, \quad s_2 = \lambda + 3a\mu, \quad t_1 = 3a\mu, \quad t_2 = \lambda,$$
$$s_1 = \lambda - 3a\mu, \quad s_2 = \lambda + a\mu, \quad t_1 = 9a\mu, \quad t_2 = \lambda.$$

From the manner of formation of the expressions for x^2 and y^2, it is easy to see that as many substitutions as desired may be found such that one will be a perfect square and the other will contain λ^2 with coefficient unity.

The following taken in order are examples illustrative of the above methods:

$$a = 2, \quad m = 1, \quad n = 1, \quad \lambda = -5, \quad \mu = 3: \quad 11^4 + 2 \cdot 8^4 = 109^2 + 2 \cdot 74^2,$$

$$a = 2, \quad m = 1, \quad n = 1, \quad \lambda = 13, \quad \mu = -6: \quad 19^4 + 2 \cdot 25^4 = 263^2 + 2 \cdot 649^2.$$

§ 5. Case when $b = \sigma^2 a$. The equation $x^4 + ay^4 = u^2 + \sigma^2 av^2$. In order to develop the theory for this case we shall treat the more general equation

$$x^4 + ay^4 = u^2 + bv^2 \tag{28}$$

by means of more general formulæ than have been used in the preceding sections. This may be done by extending either of the sets of numbers $x^4 + ay^4 - bv^2$ and $u^2 + bv^2 - ay^4$ so as to form a multiplicative domain. Extending the latter, the required set of numbers is $u^2 + bv^2 - az^2 - abw^2$.

Consider the equation

$$u^2 + bv^2 - ay^4 - abw^2 = x^4. \tag{29}$$

If $g(u, v, y, w) = u^2 + \alpha v^2 + \beta y^2 + \alpha\beta w^2$, it can readily be verified (cf. R. D. Carmichael, l. c., p. 37) that

$$\begin{aligned} \{g(u, v, y, w)\}^2 &= g(u^2 - \alpha v^2 - \beta y^2 + \alpha\beta w^2,\ 2uv - 2\beta yw,\ 2yu + 2\alpha wv,\ 0), \\ &= g(u^2 - \alpha v^2 + \beta y^2 + \alpha\beta w^2,\ 2uv,\ 2\alpha wv,\ 2yv). \end{aligned} \tag{30}$$

Hence, using the first of equations (30), we see that (29) will be satisfied if

$$\begin{aligned} u &= u_1^2 - bv_1^2 + ay_1^2 - abw_1^2, \\ v &= 2u_1v_1 + 2ay_1w_1, \\ y^2 &= 2y_1u_1 + 2bw_1v_1, \\ w &= 0, \\ x^2 &= u_1^2 + bv_1^2 - ay_1^2 - abw_1^2. \end{aligned} \tag{31}$$

Suppose first that $b = m^2a$, where m is an integer. Also, put $u_1 = m(\lambda - a\mu)$, $v_1 = 2\lambda$, $y_1 = 2m(\lambda - a\mu)$, $w_1 = 4\mu$ in (31). Then

$$\begin{aligned} u &= m^2[\lambda^2 - (8a^2 + 2a)\lambda\mu + (4a^3 - 15a^2)\mu^2], \\ v &= 4m(\lambda^2 + 3a\lambda\mu - 4a^2\mu^2), \\ y &= 2m(\lambda + a\mu), \\ x^2 &= m^2[\lambda^2 + (8a^2 - 2a)\lambda\mu - (4a^3 + 15a^2)\mu^2]. \end{aligned} \tag{32}$$

From (24) we see that the last of equations (32) is satisfied if

$$\begin{aligned} x &= m[\alpha^2 + (8a^2 - 2a)\alpha\beta - (4a^3 + 15a^2)\beta^2], \\ \lambda &= \alpha^2 + (4a^3 + 15a^2)\beta^2, \\ \mu &= 2\alpha\beta + (8a^2 - 2a)\beta^2. \end{aligned} \tag{33}$$

Equations (32) and (33) afford two-parameter solutions of equations of the general type

$$x^4 + ay^4 = u^2 + m^2av^2. \tag{34}$$

Now, put $a = m^2b$, $u_1 = m(\lambda - b\mu)$, $v_1 = 2m^2\lambda$, $y_1 = 2m(\lambda - b\mu)$, $w_1 = 4\mu$ in (31). Then, after removing a factor m^2 common to x^2, y^2, u, v, we have

$$\begin{aligned} u &= \lambda^2 - (8m^2b^2 + 2b)\lambda\mu + (4m^2b^3 - 15b^2)\mu^2, \\ v &= 4m(\lambda^2 + 3b\lambda\mu - 4b^2\mu^2), \\ y &= 2(\lambda + b\mu), \\ x^2 &= \lambda^2 + (8m^2b^2 - 2b)\lambda\mu - (4m^2b^3 + 15b^2)\mu^2. \end{aligned} \tag{35}$$

Making use of (24) we see that the last of equations (35) is satisfied if

$$\begin{aligned} x &= \alpha^2 + (8m^2b^2 - 2b)\alpha\beta - (4m^2b^3 + 15b^2)\beta^2, \\ \lambda &= \alpha^2 + (4m^2b^3 + 15b^2)\beta^2, \\ \mu &= 2\alpha\beta + (8m^2b^2 - 2b)\beta^2. \end{aligned} \tag{36}$$

Equations (35) and (36) afford two-parameter solutions of equations of the general type

$$x^4 + m^2by^4 = u^2 + bv^2. \tag{37}$$

If we put $m = 1$ in (32), (33) and (35), (36), we are led to solutions of (21). These are identical.

As examples illustrating the use of the above methods, we have the following:

$b = m^2a,\ a = 1,\ m = 2,\ b = 4,\ \alpha = 2,\ \beta = 1$: $1^4 + 1\cdot 22^4 = 319^2 + 4\cdot 182^2$,

$a = m^2b,\ b = 1,\ m = 2,\ a = 4,\ \alpha = 2,\ \beta = 1$: $11^4 + 4\cdot 46^4 = 4231^2 + 1\cdot 152^2$.

§ 6. The general case. The equation $x^4 + ay^4 = u^2 + bv^2$. For the purpose of developing the theory for the general case we may employ equations (31). Put $y_1 = 0,\ v_1 = 2bw_1$ in (31). Then

$$\begin{aligned} u &= u_1^2 - (4b^3 + ab)w_1^2, \\ v &= 4bu_1w_1, \\ y &= 2bw_1, \\ x^2 &= u_1^2 + (4b^3 - ab)w_1^2. \end{aligned} \tag{38}$$

The last equation (38) is satisfied if

$$\begin{aligned} x &= u_2^2 + (4b^3 - ab)w_2^2, \\ u_1 &= u_2^2 - (4b^3 - ab)w_2^2, \\ w_1 &= 2u_2w_2. \end{aligned} \tag{39}$$

Equations (38) and (39) clearly afford a solution of (28).

Now, put $y_1 = 0,\ w_1 = 2bv_1$ in equations (31). They then become

$$\begin{aligned} u &= u_1^2 - (b + 4ab^3)v_1^2, \\ v &= 2u_1v_1, \\ y &= 2bv_1, \\ x^2 &= u_1^2 + (b - 4ab^3)v_1^2. \end{aligned} \tag{40}$$

The last of equations (40) will be satisfied if we write

$$\begin{aligned} x &= u_2^2 + (b - 4ab^3)v_2^2, \\ u_1 &= u_2^2 - (b - 4ab^3)v_2^2, \\ v_1 &= 2u_2v_2. \end{aligned} \tag{41}$$

Equations (40) and (41) afford a solution of (28).

Again, using the second formula (30), the last of equations (31) is satisfied if

$$\begin{aligned} x &= u_2^2 + bv_2^2 - ay_2^2 - abw_2^2, \\ u_1 &= u_2^2 - bv_2^2 - ay_2^2 - abw_2^2, \\ v_1 &= 2u_2v_2, \\ y_1 &= 2bw_2v_2, \\ w_1 &= 2y_2v_2. \end{aligned} \tag{42}$$

Put $y_2 = w_2 = bv_2$. Then, from the third equation in (31), we have the further restriction

$$y^2 = 4b^2v_2^2[u_2^2 + 2u_2v_2 - (b + ab^2 + ab^3)v_2^2], \tag{43}$$

which is satisfied in accordance with (24) if

$$\begin{aligned} y &= 2bv_2[m^2 + 2mn - (b + ab^2 + ab^3)n^2], \\ u_2 &= m^2 + (b + ab^2 + ab^3)n^2, \\ v_2 &= 2mn + 2n^2. \end{aligned} \tag{44}$$

It will now be seen that equations (31), (42), (44) afford a solution of (28).

The following taken in order are examples illustrative of the above methods:

$a=5,\ b=2,\ u_2=1,\ w_2=1,\ u_1=-21,\ w_1=2:\quad 23^4+5\cdot 8^4 = 273^2+2\cdot 336^2,$

$a=3,\ b=2,\ u_2=1,\ v_2=1,\ u_1=95,\ v_1=2:\quad 93^4+3\cdot 8^4 = 8633^2+2\cdot 380^2,$

$a=3,\ b=2,\ n=1,\ u=1,\ u_2=39,\ v_2=4:\ 977^4+3\cdot 560^4=663457^2+2\cdot 618864^2.$

If we operate on the last of equations (31) by means of the first formula (30) and make certain easy transformations, we will be led at once to the following solution of (28):

$$\begin{aligned} x &= \alpha^4 - a\beta^4 + b\gamma^2, \\ y &= 2\alpha\beta\sqrt{\alpha^4 + a\beta^4 - b\gamma^2}, \\ u &= (\alpha^4 + a\beta^4 - b\gamma^2)^2 - 4b\alpha^4\gamma^2 + 4a\alpha^4\beta^4, \\ v &= 4\alpha^2\gamma(\alpha^4 + a\beta^4 - b\gamma^2). \end{aligned} \tag{45}$$

These formulæ have been developed in full by R. D. Carmichael (l. c., pp. 46–47) and will therefore be dismissed with this remark.

GRAPHICAL CONSTRUCTIONS FOR A FUNCTION OF A FUNCTION AND FOR A FUNCTION GIVEN BY A PAIR OF PARAMETRIC EQUATIONS.[1]

By W. H. ROEVER, Washington University.

It is easy to see that if $y = \phi(u)$ and $u = \psi(x)$ are the equations of the orthographic projections of a space curve on the planes y–O–u and u–O–x determined by the system of rectangular cartesian axes O–x, u, y, then $y = \phi[\psi(x)]$ is the equation of the orthographic projection of this space curve on the plane x–O–y (see Fig. 1).

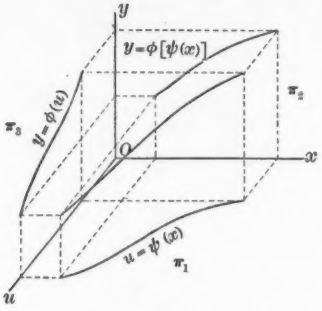

FIG. 1.

Let us now suppose the planes $\pi_1 \equiv xOu$ and $\pi_3 \equiv uOy$ to be revolved around the axes of x and y respectively until they coincide with the plane $\pi_2 \equiv yOx$, and in such a way that the revolved positions of the positive u-axis coincide with the negative portions of the unrevolved axes of x and y (see Fig. 2). If a general point of space be denoted by P, let its orthographic projections on π_1, π_2, and π_3 be denoted by $'P$, P'' and $'''P$ respectively. During the revolutions just made the point P'' remains unchanged, but the points $'P$ and $'''P$ assume the new positions P' and P''' respectively (see Fig. 2). The plane π_2, which now also contains the revolved positions of the planes π_1 and π_3, is the *drawing plane of Mongean Descriptive Geometry* and the unrevolved axes of x and y are called the *first* and *second ground lines* respectively (see Fig. 3). The internal bisector of the right angle formed by the positive portions of these ground lines is called the *line b*. From the method of obtaining the points P', P'', P''' of the drawing plane it follows that:

(1) *P' and P'' lie on the same perpendicular to the first ground line,*

[1] Presented to the American Mathematical Society, November 27, 1915.

After this paper was written my attention was called to a construction by Professor E. H. Moore which is practically identical with the one here given. Professor Moore's construction is contained under *Linkage B* in his article "Cross-section paper as a mathematical instrument" published in *The School Review*, May, 1906. Dr. A. J. Kempner has also discovered a construction which includes the one here given. His paper will be published in a later number of this MONTHLY.

(2) *P'' and P''' lie on the same perpendicular to the second ground line,*

(3) *lines through P' and P''' parallel respectively to the first and second ground lines, meet in a point P° of the line b* (see Fig. 3). It is important to observe that this is true regardless of the position of P in space; that is, P need not necessarily lie in the first octant as shown in Fig. 2. Following general usage we shall

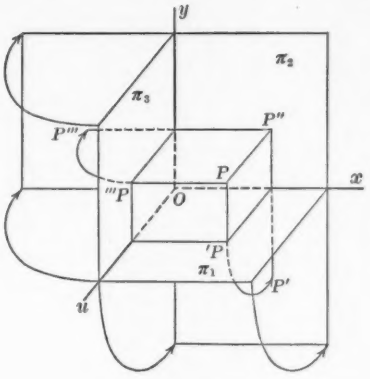

FIG. 2.

call the points P', P'', P''' the *projections in the drawing plane* of the point P of space. We may then say that *no matter what the position of a point in space may be, its projections in the drawing plane are three vertices of a rectangle whose sides are parallel to the ground lines, and whose fourth vertex lies on the line b.*

By combining the facts stated in the two preceding paragraphs we are led to the *Construction* (Fig. 3). Assume a pair of perpendicular lines (one horizontal and the other vertical) intersecting in a point O. Let these represent respectively the axes of x and y, x being positive to the right of O and y positive above O. Let each of these lines also represent the axis of u, in which however the positive and negative directions are the reverse of those in x and y. Then draw through O the line b bisecting internally the angle formed by the positive portions of the axes of x and y. Now plot with respect to the axes of x and u the graph of $u = \psi(x)$ and with respect to u and y the graph of $y = \phi(u)$. All this being done, we find the required graph of $y = \phi[\psi(x)]$ by drawing through each point P° of the line b a horizontal line cutting $u = \psi(x)$ in P' and a vertical line cutting $y = \phi(u)$ in P''', and then through P' a vertical line and through P''' a horizontal line. The last two lines intersect in a point P'' of the required graph.

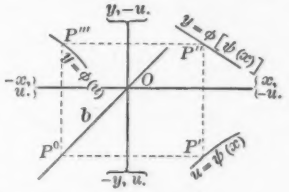

FIG. 3.

In Fig. 4, this construction is used to find the graph of $y = e^{1/x}$ from the graphs of $y = e^u$ and $u = 1/x$.

This method also enables one to construct a curve which is given by a pair of parametric equations

$$x = f(u), \qquad y = \phi(u).$$

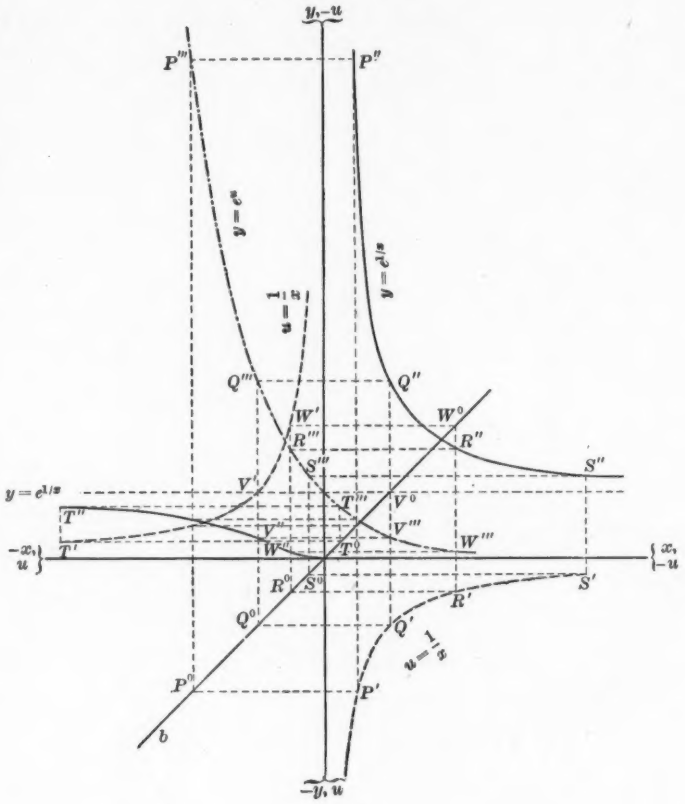

FIG. 4.

For the problem of eliminating u between these equations is the same as that of finding the function $y = \phi(u)$ of the function $u = \psi(x)$, where $u = \psi(x)$ is obtained by solving $x = f(u)$ for u in terms of x. In Fig. 5, the method is used to construct the curve of which the parametric equations are

$$x = u^2, \qquad y = u^3.$$

It is interesting to note that for $u = 0$ both of the functions $f(u) = u^2$, $\phi(u) = u^3$ have vanishing derivatives and hence the curve $y^2 = x^3$ has a singularity—in this case a cusp—at the origin. The curves in Figs. 4 and 5 have been accurately

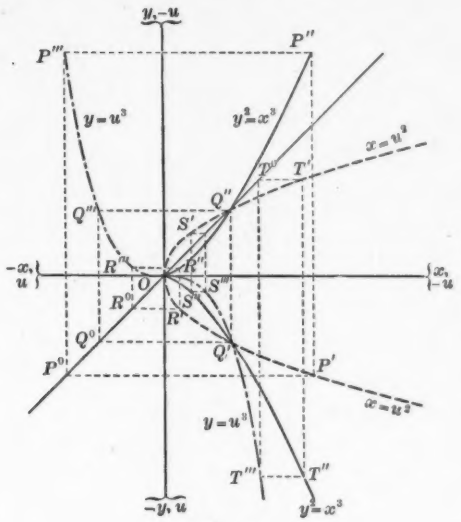

FIG. 5.

constructed. However, the construction under consideration often enables one to determine the salient features of the graph of $y = \phi[\psi(x)]$ from a mere rough plotting of the graphs of $y = \phi(u)$ and $u = \psi(x)$.[1]

A NOTE ON THE SUM OF THE REMAINDERS OF A SERIES.

By GLENN JAMES, Purdue University.

Suppose the series

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

has the sum S, and consider the series formed from its remainders, which is

$$S + [S - a_0] + [S - (a_0 + a_1)] + \cdots + [S - (a_0 + a_1 + \cdots a_{n-1})] + \cdots \tag{2}$$

or

$$R_0 + R_1 + R_2 + \cdots + R_n + \cdots. \tag{3}$$

The assumption that (1) converges is equivalent to the assumption that

$$\lim_{n=\infty} R_n = 0.$$

Moreover, any convergent or divergent series

$$H_0 + H_1 + H_2 + \cdots + H_n + \cdots,$$

[1] It might be remarked that the addition of the 45° line b on ordinary rectangular plotting paper would make of that paper a splendid medium for carrying out the above construction.

for which $\lim_{n=\infty} H_n = 0$, is the remainder series of

$$(H_0 - H_1) + (H_1 - H_2) + \cdots + (H_{n-1} - H_n) + \cdots,$$

which converges since its sum is H_0.

Hence, there is an infinite class of convergent series whose remainder series converge and an infinite class whose remainder series diverge. The following theorems determine, upon the basis of the simpler convergence tests, certain types of series which belong to the former class.

THEOREM I. *Any series whose terms alternate in sign, never increase in absolute value, and tend to zero, possesses an absolutely convergent remainder series if the original series converges absolutely.*

Given an absolutely convergent series,

$$a_0 - a_1 + a_2 - a_3 + \cdots, \tag{4}$$

where

$$a_0 \geqq a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq 0 \qquad \text{and} \qquad \lim_{n=\infty} a_n = 0.$$

The sum of such a series lies between zero and the first term, and the nature of such a series is not altered by dropping any number of consecutive terms from the beginning of the series. Also $S - S_{n-1}$, $(= R_n)$, is the sum of (4) after the $(n-1)$th term. Hence,

$$|R_n| < |a_n|, \qquad n = 0, 1, 2, 3, \cdots.$$

By hypothesis $\sum_0^\infty a_n$ is absolutely convergent. Therefore, $\sum_0^\infty R_n$ is absolutely convergent.

The following example shows that a series whose terms alternate in sign, continually decrease numerically, and tend to zero, does not necessarily have a convergent remainder series:

$$S = (\tfrac{1}{3} + \tfrac{2}{4}) - (\tfrac{1}{3} + \tfrac{2}{8}) + (\tfrac{1}{4} + \tfrac{2}{16}) - (\tfrac{1}{4} + \tfrac{2}{32}) + (\tfrac{1}{5} + \tfrac{2}{64}) - (\tfrac{1}{5} + \tfrac{2}{128}) \cdots = \tfrac{1}{2}. \tag{5}$$

This series does not converge absolutely, for each term is greater in absolute value than the corresponding term of the divergent series $\sum_3^\infty (1/n)$. Noting that $R_1 = R_0 - a_0$, $R_2 = R_1 - a_1$, $\cdots$, the remainder series is evidently

$$\tfrac{1}{2} - (\tfrac{1}{3} + \tfrac{1}{4}) + \tfrac{1}{8} - (\tfrac{1}{4} + \tfrac{1}{16}) + \tfrac{1}{32} - (\tfrac{1}{5} + \tfrac{1}{64}) + \tfrac{1}{128} - \cdots$$

or

$$\tfrac{1}{2} - (\tfrac{1}{3} + \tfrac{1}{8}) - (\tfrac{1}{4} + \tfrac{1}{32}) - (\tfrac{1}{5} + \tfrac{1}{128}) - \cdots,$$

which plainly diverges.

THEOREM II. *If a series converges by the ratio test, its remainder series converges.*

Consider first a series $\sum_0^\infty a_n$ for which $\lim_{n=\infty} |a_n/a_{n-1}| = \alpha < 1$. Then for any

positive ϵ there exists an n' such that

$$\alpha + \epsilon > \left|\frac{a_n}{a_{n-1}}\right| > \alpha - \epsilon, \qquad n \geqq n'.$$

Hence to every $n' + i$ there corresponds an ϵ_i, $|\epsilon_i| < 2\epsilon$, $(i = 1, 2, 3, \cdots)$, such that

$$\left|\frac{a_{n'}}{a_{n'-1}}\right| = \left|\frac{a_{n'+1}}{a_{n'}}\right| + \epsilon_1 = \left|\frac{a_{n'+2}}{a_{n'+1}}\right| + \epsilon_2 = \cdots.$$

Let the ratio $|a_{n'}/a_{n'-1}|$ be denoted by $K_{n'}$, and collect terms. Then

$$K_{n'} = \left|\frac{a_{n'}}{a_{n'-1}}\right| = \frac{|a_{n'+1}| + \epsilon_1 |a_{n'}|}{|a_{n'}|} = \frac{|a_{n'+2}| + \epsilon_2 |a_{n'+1}|}{|a_{n'+1}|} = \cdots. \tag{6}$$

Whence

$$\begin{aligned} K_{n'}|a_{n'-1}| &= |a_{n'}|, \\ K_{n'}|a_{n'}| &= |a_{n'+1}| + \epsilon_1 |a_{n'}|, \\ K_{n'}|a_{n'+1}| &= |a_{n'+2}| + \epsilon_2 |a_{n'+1}|, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Adding these identities, we have

$$K_{n'}[|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots] = [|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots] + [\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots].$$

Whence, making use of (6),

$$K_{n'} = \frac{|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} + \frac{\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} = \left|\frac{a_{n'}}{a_{n'-1}}\right|. \tag{7}$$

Now

$$|[\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots |] < 2\epsilon[|a_{n'}| + |a_{n'+1}| + \cdots] < 2\epsilon[|a_{n'-1}| + |a_{n'}| + \cdots].$$

Hence

$$\left|\frac{\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots}\right| < 2\epsilon.$$

But $\lim_{n'=\infty} 2\epsilon = 0$, which gives

$$\lim_{n'=\infty} K_{n'} = \lim_{n'=\infty} \frac{|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} = \lim_{n'=\infty} \left|\frac{a_{n'}}{a_{n'-1}}\right| = \alpha. \tag{8}$$

Let $R_{n'} = |a_n| + |a_{n+1}| + \cdots$, and $R_n = a_n + a_{n+1} + \cdots$. Then $\sum_0^\infty R_{n'}$ converges by (8). But $|R_n| \leqq R_{n'}$. Therefore $\sum_0^\infty R_n$ converges.

Consider now a series $\sum_0^\infty a_n$ for which the ratio a_n/a_{n-1} approaches no limit but remains less than α which is less than unity for all values of n greater than n'.

Then the series $\sum_0^\infty a_n$ and its remainder series $\sum_0^\infty R_n$ are dominated respectively by the series

$$\sum_0^\infty A_n = \sum_0^{n'-1} |a_n| + \sum_{m=0}^\infty |a_{n'}| \alpha^m \tag{9}$$

and its remainder series $\sum_0^\infty \bar{R}_n$. But (9) is a series to which the above proof applies. Hence $\sum_0^\infty \bar{R}_n$ and consequently $\sum_0^\infty R_n$ converge.

COROLLARY. *If the ratio of the nth to the $(n-1)$th term of a convergent series of positive terms has a limit, the corresponding ratio for the remainder series has the same limit.*

If $\sum_0^\infty a_n$ converges, the first part of the proof of the above theorem does not require that α, $(= \lim_{n=\infty} |a_n/a_{n-1}|)$, be less than unity. The corollary follows, then, from (8).

Examples (a), (b) and (c), below, illustrate the preceding theorem and corollary.

(a) $$S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2.$$

The remainder series is

$$S' = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 3.$$

(b) The geometric series

(i) $$\sum_0^\infty ar^n = \frac{a}{1-r}, \qquad |r| < 1,$$

has the remainder series

(ii) $$\sum_{n=0}^\infty [ar^n + ar^{n+1} + \cdots] = \sum_0^\infty \frac{ar^n}{1-r} = \frac{a}{(1-r)^2}.$$

The remainder series of (ii) is

(iii) $$\sum_0^\infty \frac{ar^n}{(1-r)^2} = \frac{a}{(1-r)^3}.$$

Thus, the mth remainder series of (i) is

$$\sum_{n=0}^\infty \frac{ar^n}{(1-r)^m} = \frac{a}{(1-r)^{m+1}}, \qquad |r| < 1.$$

Hence the sums of the successive remainder series increase indefinitely when $1 > r > 0$ and approach zero when $0 > r > -1$.

(c) $$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots.$$

The remainder series is

$$S' = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

whence

$$\lim_{n=\infty} \frac{a_n}{a_{n-1}} = L\frac{n-1}{n+1} = L\frac{R_n}{R_{n-1}} \doteq L\frac{n-1}{n} = 1.$$

A further classification of series, as to the convergence or divergence of their remainder series, might be made upon the basis of other convergence tests. And in the case of series with variable terms it would be of interest to study such properties as uniform convergence with reference to their remainder series.

BOOK REVIEWS.

Analytic Mechanics. By JOHN ANTHONY MILLER, Ph.D., Professor of Mathematics, Swarthmore College, and SCOTT BARRETT LILLY, C.E., Assistant Professor of Engineering, Swarthmore College. D. C. Heath and Company, 1915.

The motive of the authors of this book is described in the preface in the following words: "We have attempted to write a rigorous, teachable introduction to the study of mechanics. We believe that certain fundamental principles of mechanics, used in common by students in the various branches of engineering, in theoretical physics, and in celestial mechanics are essential to the satisfactory progress of a student in any of these fields. To this end, we have chosen as our subject matter only such fundamental theorems." The result is a book of 292 pages, consisting of an introduction and sixteen chapters treating of the following topics: Composition and resolution of forces acting on a particle; statics of a particle; forces acting on a rigid body; vectors; statics of a rigid body; center of gravity; friction; flexible cords; kinetics of a particle; motion of a particle in a plane curve; work and energy; dynamics of a rigid body; kinetic friction. The difficult problem of choice of material has been solved with sufficient success to produce a book without important omissions and of usable dimensions. A number of excellent treatises on mechanics have been published in the English language during the past generation, but the best of these have been unadapted for use as introductory textbooks because of their unmanageable dimensions. Of the numerous attempts which have been made to meet the demand for a satisfactory introductory course, the book of Miller and Lilly is among the most successful in the important matter of selection of material.

The authors have also attained a considerable degree of success in their attempt "always to be rigorous" while producing "a book that is distinctly teachable." Teachable it undoubtedly is; explanations are stated in simple, direct language; too much is not presupposed in the way of mathematical knowledge and acumen on the part of the student. The exercises for the student are for the most part sufficiently simple and designed to illustrate the principles of mechanics rather than the elegance of a mathematical method. It is mainly in the explanation of principles, rather than in the application to particular cases, that there occur what the reviewer regards as lapses of rigor.

The treatment shows the sympathy for practical applications which would be expected from the fact that one of the authors is a teacher of engineering. Engineering applications are however brought in incidentally rather than by formal separation in the treatment; thus the determination of stresses in jointed frames is explained in the chapters on statics of a particle and statics of a rigid body.

The laws of motion are stated in the language of Newton (the English rendering being nearly identical with that of Thomson and Tait), and very little is said by way of explaining the meaning of the laws. The equation of motion

for a particle is habitually employed in the form $F = (W/g)a$; but this is deduced from the equation $F = kma$ which is given as the algebraic expression of the second law and is thus implicitly taken as the "fundamental" equation. The explanation of basic principles may perhaps fairly be characterized as traditional; that is, it is in the main the method which has predominated in English and American books since the publication of Thomson and Tait's *Natural Philosophy*. To a considerable extent the lapses of rigor referred to above are characteristic of this traditional method. It seems worth while to refer in some detail to certain of the defects which, according to the view of the present writer, have been perpetuated in this traditional treatment. These defects are associated with the following subjects: (1) The concept of force, (2) the composition of velocities and accelerations, (3) the theory of work and energy.

1. The vagueness of the definition of force is perhaps the most serious defect in the traditional treatment. A proper explanation should wholly remove this vagueness. The practical concept of force involves the following points: (*a*) A force is a push or a pull exerted upon a body or portion of matter; (*b*) every force acting on one body is exerted by some other body (*i. e.*, there is some body which does the pushing or pulling); (*c*) when a body A is pulling or pushing a body B, the body B is at the same time pulling or pushing A equally in the opposite direction and along the same line; (*d*) a force tends to change the velocity of the body upon which it acts. The traditional definition and explanation usually give no explicit statement of either (*b*) or (*c*)—often scarcely a hint of these important facts—and in many cases there is no appeal even to the notion of push or pull in explaining force.[1]

The book under review shares the traditional vagueness in the explanation of force. At the beginning of Chapter I we read: "Force is that which changes, or tends to change, the motion of matter. It is either a push or a pull." There is no hint here or in the context that two bodies or portions of matter are concerned, nor any intimation that the real meaning of the law of action and reaction is the above statement (*c*).[2] Moreover the explanation is marred by statements which are not merely vague but incorrect: "That which moves matter is *force*" (p. 4); "In a general way we have known that if a body is moving it is acted upon by some force" (p. 6).

The vagueness which results from a failure to keep definitely in mind the points above designated as (*b*) and (*c*) appears in the treatment of internal and external forces in the dynamics of a rigid body (Art. 249). There is no clear

[1] An examination of six books taken at random among textbooks on mechanics published during the past ten years failed to find in any one the use of the words push or pull to explain force; and it is fair to say that the majority of books either give no hint at all that two bodies are concerned in the action of every force or refer to it merely incidentally. Undoubtedly many students have completed a course in mechanics without having their attention called either to this fact or to the real meaning of the law of action and reaction.

[2] Some textbook writers appear to take it for granted that the explicit statement (*c*) is unnecessary because sufficiently implied by the statement "to every action there is an equal and contrary reaction." Nothing is more certain, however, than that able writers have repeatedly been led into mistakes by the failure to recognize that (*c*) is the real meaning of the third law.

statement of the essential distinction between these two classes of forces (that an internal force is exerted upon some part of the body whose motion is being studied *by another part of the same body*, while an external force is exerted *by some other body*). When this is clearly stated, and when the law of action and reaction is understood to have the meaning (*c*), it is at once seen that the internal forces cancel out of the equations obtained by summing the equations of motion written for all individual particles. This cancellation depends merely upon the fact that, for any pair of forces constituting action and reaction, the vector sum is zero and the sum of the moments about any axis is zero; a serious confusion of language, if not of thought, is involved in the statement that " the internal forces are in equilibrium among themselves."[1]

2. The vagueness of the traditional treatment of the composition of velocities is well illustrated by the words " if a particle is subject to two simultaneous velocities " (p. 158 of the book under review). It is true that such language is very common and that there is a stock explanation of its meaning. This explanation refers to a particle moving with a certain velocity with respect to a body which is itself in motion. Thus if a particle is supposed to " move along a rod with uniform speed " while the rod is " carried parallel to itself " with uniform velocity, it is easily seen that the actual velocity of the particle is the vector sum of the velocity of the rod and the velocity of the particle with respect to the rod. There is no vagueness here; but the explanation throws no light whatever upon the meaning of simultaneous velocities *with respect to a single base or body of reference*. Yet this is precisely what needs explanation, since the language commonly employed implies that a body may have at the same time two different velocities *with respect to the same base*. The fact is that, so long as motion is referred to one definite base, a particle has at any instant one definite velocity. This actual velocity, being a directed or vector quantity,[2] may *for the purposes of mathematical analysis or computation* be resolved into components by the rules of vector addition; but this does not at all warrant the statement that a particle has (or is " subject to ") two or more different velocities at the same time.[3]

What has just been said of velocity applies also to acceleration, so far as the meaning of vector composition and resolution is concerned. An acceleration is

[1] The statement that an action and reaction constitute a pair of counterbalancing forces is one of the most unfortunate errors which are perpetuated by the carelessness of able writers. Action and reaction never counterbalance each other; they act on different portions of matter.

[2] The fact that " velocity is a vector " is not a proposition requiring proof (as might be inferred from the language used on p. 159); the very conception of the instantaneous motion of a particle makes its velocity a directed magnitude.

[3] The logical point involved may be illustrated by an analogy. The position of a particle may be specified by a vector OA, drawn from a fixed origin O to the instantaneous position A, and this position-vector may be expressed as the sum of two vectors OB, OC so taken that $OBAC$ is a parallelogram. If we were to interpret this as meaning that the particle at A may be regarded as occupying simultaneously the positions B and C, we should be making the same kind of statement that is made when it is said that a particle has at the same instant two different velocities.

It may be remarked also that the use of the words " subject to a velocity " seems to indicate a confusion of thought, since they imply that the continuance of a velocity must be due to some external cause—a notion explicitly negatived by the first law of motion.

a vector quantity and may, for the purposes of analysis or computation, be resolved into components; but this does not warrant the statement that a particle has at the same time two different accelerations with respect to one given base. *Apart from the idea of force* the resolution of the actual acceleration into vector components is an arbitrary geometric process. When we come to interpret the second law of motion as applied to a particle acted upon by more than one force, there is of course a special reason for treating the acceleration as made up of vector components, each associated with a particular force; the explanation of this fact is however not a proof that " acceleration is a vector " but is an explanation of the law of composition of forces. The idea that acceleration is a directed or vector quantity is purely kinematical—quite independent of any law involving force; in fact the full understanding of Newton's second law must presuppose the notion of acceleration defined as the rate of change of the velocity-vector.

3. In the traditional treatment there is a logical confusion associated with the definitions of work and energy. Energy is defined as *the ability of a body to do work*, while work is defined as being done not *by a body* but *by a force*.

In the book under review the chapter on Work and Energy begins with an admirably simple and clear elementary treatment of work done by a force. Kinetic energy is then defined by an algebraic formula, and a proof is given of the theorem that " the change in the kinetic energy of a particle equals the work done by the resultant force acting on it." Then follows a brief discussion of the condition for the existence of a force function U, leading to the definition of potential energy as " the negative of the function U." Thus far there is no lack of logical rigor. There follows however an Article headed " The Energy of a Particle is its Ability to do Work " in which we read: " It seems reasonable from the Third Law of Motion that the equations[1] . . . may be read the other way, viz., that in changing from the velocity v to the velocity v_0 the particle will do work on another body equal to w. This comes to saying that a particle will do as much work in giving up its velocity as has been done on the particle in order that it might acquire it." But what is meant by a particle " doing work upon a body "? This question is nowhere answered. The passage quoted is in fact fairly representative of the logical vagueness of the traditional treatment. To remove this vagueness it is necessary to adopt a clear *definition* of " work done by a body," from which it may be *proved* that the " ability of a particle to do work " (*i. e.*, the quantity of work it will do in coming to rest) is equal to $\frac{1}{2}mv^2$. With reference to the above quotation it is also to be said that the citation of the third law of motion is quite irrelevant and encourages confusion as to the meaning of that law.

The space occupied by the foregoing criticisms is out of proportion to their applicability to the book under review; the criticisms are in fact directed rather at the traditional treatment than at the example of it presented by this book. The usefulness of a book as a text depends far more upon the treatment of concrete

[1] The equations referred to express the fact that " the change of the kinetic energy of a particle equals the work done by the forces acting on it."

applications than upon the logical rigor with which fundamental principles are established. The book of Miller and Lilly seems to be decidedly usable as a class textbook, and is likely to find favor among teachers.

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PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

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PROBLEMS FOR SOLUTION.

ALGEBRA.

468. Proposed by H. C. FEEMSTER, York College, Nebraska.

In each of the following series find the nth term and sum:

(a) $2 + 5 + 9 + 15 + 24 + \cdots$

(b) $1 + 6 + 10 + 20 + 35 + \cdots$

(c) $1 + 5 + 15 + 35 + 70 + \cdots$

469. Proposed by T. H. GRONWALL, New York City.

Show that the equation

$$f(x) = 2ax^4 + (1 - b)x^3 + b(1 - b)x - 2ab = 0,$$

where $0 < b < 1$, $a > 0$ and $a^2 > b$ has only one positive root and that this root lies between the roots of $g(x) = x^2 - 2ax + b = 0$.

470. Proposed by ERNEST W. BROWN, Yale University.

There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, where s is an integral multiple of $\frac{1}{2}$?

GEOMETRY.

499. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

500. Proposed by R. T. McGREGOR, Bangor, California.

$OABC$, $OA'B'C'$ are two straight lines such that AA', BB', CC' are parallel. AB', $A'C$ meet in P; $A'B$ and AC' meet in Q. Show by synthetic projective geometry that PQ is parallel to AA'. MILNE'S *Projective Geometry*, Chap. I, Ex. 20.

501. Proposed by R. P. BAKER, University of Iowa.

Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box.

502. Proposed by R. P. BAKER, University of Iowa.

A designer of machinery requires a curve having the following properties:

(1) A closed curve touching a given circle at two diametral points and enclosing it.

(2) The sum of the three radii from the center of this circle to the curve which make with each other angles of 120° is constant.

(3) The locus of a point which lies at some constant distance from the curve on its inner normal must be such that it is also the locus of a point fixed on a bar of some simple linkage. In estimating the value of the word "simple" pivoted bars are preferred to slides and the total number should be as small as possible.

Condition (3) is needed to enable a cylinder to be ground accurately to the curve.

CALCULUS.

417. Proposed by H. S. UHLER, Yale University.

To the degree of approximation indicated, show that

$$(\sqrt{-1})^{\sqrt{-1}} = 0.207,879,576,351.$$

418. Proposed by B. F. FINKEL, Drury College.

A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yards wide along each straightaway is to be bought for grandstands, training quarters, etc. If the land costs \$200 an acre, what will be the least possible cost of the land required?

GRANVILLE'S *Differential and Integral Calculus*, p. 116.

Is there anything wrong with this problem? Explain the contradiction involved in the solution.

MECHANICS.

334. Proposed by HORACE OLSON, Chicago, Illinois.

A particle of elasticity e is projected with a velocity v at an angle ϕ with a plane inclined to the horizontal at an angle ψ; its plane of motion is perpendicular to the inclined plane. Show that after $2v \sin \phi/g(1 - e) \cos \psi$ seconds it will cease to rebound and will move along the plane with an initial velocity $v \cos \phi - 2v \sin \phi \tan \psi/(1 - e)$ and a uniform acceleration, $g \sin \psi$, down the plane.

(This problem is a generalization of problem 289 in Mechanics, which appeared in the March, 1914, issue of the MONTHLY.)

335. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

A heavy particle is projected upwards with a velocity V in a medium resisting as the nth power of the velocity. Prove that the elevation of the particle when the velocity downwards is V is equal to LT where L is the limiting velocity and T is the time in which the particle falling from rest in the medium will acquire a velocity V^2/L.

NUMBER THEORY.

254. Proposed by HORACE OLSON, Chicago, Illinois.

Find three integers, x, y, and z, such that $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$, and $x^2 + y^2 + z^2$ are all perfect squares.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

456. Proposed by PAUL CAPRON, U. S. Naval Academy.

If

$$S_{i,n} = \sum_{k=1}^{k=n-i+1} \frac{(i+k-1)!}{(k-1)!},$$

show that $S_{i,n}$ is equal to $1/(i+1)$ times the last term of $S_{i+1,n+1}$; as, for instance, that

$$S_{1,n} = 1 + 2 + \cdots + n = \frac{n}{2}(n+1),$$

that

$$S_{2,n} = 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \tfrac{1}{3}(n-1)n(n+1),$$

etc.

I. Solution by Howard C. Feemster, York College, Nebraska.

$$S_{1,n} = \frac{\lfloor 1}{\lfloor 0} + \frac{\lfloor 2}{\lfloor 1} + \cdots + \frac{\lfloor n}{\lfloor n-1} = 1 + 2 + \cdots + n = \frac{n}{2}(n+1);$$

$$S_{2,n} = \frac{\lfloor 2}{\lfloor 0} + \frac{\lfloor 3}{\lfloor 1} + \cdots + \frac{\lfloor n}{\lfloor n-2} = 1\cdot 2 + 2\cdot 3 + \cdots + (n-1)n = \tfrac{1}{3}n(n-1)(n+1),$$

for $n-1$ terms; and

$$S_{3,n} = \frac{\lfloor 3}{\lfloor 0} + \frac{\lfloor 4}{\lfloor 1} + \cdots + \frac{\lfloor n}{\lfloor n-3} = 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + \frac{(n-2)(n-1)n(n+1)}{4},$$

for $n-2$ terms.

Now assume

$$S_{i,n} = \frac{\lfloor i}{\lfloor 0} + \frac{\lfloor i+1}{\lfloor 1} + \frac{\lfloor 1+3}{\lfloor 2} + \cdots + \frac{\lfloor n}{\lfloor n-i} = 1\cdot 2\cdot 3\cdots i + 2\cdot 3\cdots(i+1)$$

$$+ 3\cdot 4\cdots(i+2) + \cdots + (n-i+1)(n-i+2)\cdots n \tag{1}$$

$$= \frac{(n-i+1)(n-i+2)\cdots(n)(n+1)}{i+1},$$

for $n-i+1$ terms.

But the formula is true for $n-i+1=1$ and $n-i+1=2$; for

$$\frac{\lfloor i}{\lfloor 0} = \frac{1}{i+1}\cdot\frac{\lfloor i+1}{\lfloor 1} \quad \text{and} \quad \frac{\lfloor i}{\lfloor 0} + \frac{\lfloor i+1}{\lfloor 1} = \frac{\lfloor i+2}{i+1}.$$

If (1) is true

$$S_{i,n+1} = \frac{(n-i+1)(n-i+2)\cdots(n)(n+1)}{i+1} + (n-i+2)(n-i+3)\cdots n(n+1)$$

$$= \frac{(n-i+1)(n-i+2)\cdots n(n+1) + (i+1)(n-i+2)(n-i+3)\cdots(n)(n+1)}{i+1}$$

$$= \frac{(n-i+2)(n-i+3)\cdots(n)(n+1)(n+2)}{i+1},$$

which is of the same form as (1) when the number of terms is increased by unity, and hence the proposition is true in general.

II. Solution by the Proposer.

Let $\phi(x) \equiv a^n + a^{n-1}x + a^{n-2}x^2 + \cdots + a^{n-i}x^i + \cdots + x^n$;
then

$$\frac{d^i\phi(x)}{dx^i} = \phi^{(i)}(x) \equiv i!a^{n-i} + \frac{(i+1)!}{1!}a^{n-i-1}x + \frac{(i+2)!}{2!}a^{n-i-2}x^2 + \cdots + \frac{n!}{(n-1)!}x^{n-i};$$

and

$$\phi(a) = na^n, \qquad \phi^{(i)}(a) = S_{i,n}a^{n-i}, \qquad \phi^{(n)}(a) = S_{n,n}a^0 = n!.$$

Let $f(x) \equiv (x-a)\phi(x) \equiv x^{n+1} - a^{n+1}$; then $f(a) = 0$,

$$f'(x) \equiv (x-a)\phi'(x) + \phi(x); \qquad f'(a) = \phi(a) = na^n.$$

$$f''(x) \equiv (x-a)\phi''(x) + 2\phi'(x); \qquad f''(a) = 2\phi'(a) = 2S_{1,n}\cdot a^{n-1}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{(i+1)}(x) \equiv (x-a)\phi^{(i+1)}(x) + (i+1)\phi^{(i)}(x); \qquad f^{(i+1)}(a) = (i+1)\phi^{(i)}(a) = (i+1)S_{i,n}\cdot a^{n-i}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{(n+1)}(a) = (n+1)\phi^{(n)}(a) = (n+1)!$$

Since $f(a+x) \equiv (a+x)^{n+1} - a^{n+1} \equiv \binom{n+1}{1}a^n x + \binom{n+1}{2}a^{n-1}x^2 + \cdots + \binom{n+1}{i+1}a^{n-i}x^{i+1} + \cdots + x^{n+1}$, and again,

$$f(a+x) \equiv f(a) + xf'(a) + \frac{x^2}{\lfloor 2}f''(a) + \cdots + \frac{x^{i+1}}{(i+1)!}f^{(i+1)}(a) + \cdots + \frac{x^{n+1}}{(n+1)!}(n+1)!,$$

we have, on comparison of coefficients,

$$\frac{1}{(i+1)!}f^{(i+1)}(a) = \binom{n+1}{i+1}a^{n-1} = \frac{(n+1)!}{(i+1)!(n-i)!}a^{n-1},$$

$$f^{(i+1)}(a) = (i+1)\phi^{(i)}(a) = (i+1)S_{i,n}\cdot a^{n-1} = \frac{(n+1)!}{(n-i)!}a^{n-1},$$

so that

$$S_{i,n} = \frac{1}{i+1}\cdot\frac{(n+1)!}{(n-i)!},$$

and $(n+1)!/(n-i)!$ is the last term of $S_{i+1,n+1} = \sum_{k=1}^{k=n-i+1}(i+k)!/(k-1)!$.

Also solved by HORACE OLSON.

457. Proposed by FRANK IRWIN, University of California.

If a be any number prime to m and m/a be developed as a continued fraction,

$$\frac{m}{a} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k},$$

with $a_1 \neq 0$, then there will exist a number b such that

$$\frac{m}{b} = a_k + \frac{1}{a_{k-1}} + \cdots + \frac{1}{a_2} + \frac{1}{a_1}.$$

Show that $ab \equiv \pm 1 \pmod{m}$ and determine the sign.

SOLUTION BY THE PROPOSER.

Write m and a as functions of the a's: $m = [a_1, a_2, \cdots a_k]$, $a = [a_2, a_3, \cdots a_k]$, using the *Gaussische Klammer* notation, see BACHMANN, *Niedere Zahlentheorie*, vol. 1, p. 104; or, with a different symbol, $K(a_1, a_2, \cdots a_k)$, CHRYSTAL'S *Algebra*, part II, 2d ed., p. 495.

Similarly, $b = [a_{k-1}, \cdots a_2, a_1]$, and if p_{k-1}/q_{k-1} be the next to last convergent to m/a, $p_{k-1} = [a_1, a_2, \cdots a_{k-1}]$. But, by an elementary property of these expressions, $[a_1, a_2, \cdots a_{k-1}] = [a_{k-1}, \cdots a_2, a_1)$; that is, $p_{k-1} = b$.

Again, since p_{k-1}/q_{k-1}, and m/a are successive convergents, $mq_{k-1} - ap_{k-1} = (-1)^k$; that is, $mq_{k-1} - ab = (-1)^k$, or $ab \equiv (-1)^{k-1} \pmod{m}$.

GEOMETRY.

483. Proposed by LAENAS G. WELD, Pullman, Illinois.

A circle is inscribed in a triangle. In each of the three spandrils exterior to the circle another circle is inscribed; in the remaining spandrils three other circles; and so on ad infinitum. Show that the sum of the areas of these circles is given by the formula:

$$\Sigma = \frac{\pi}{4}\cdot\frac{\Delta^2}{S^2}\left[\frac{1}{\sin(A/2)} + \frac{1}{\sin(B/2)} + \frac{1}{\sin(C/2)} - 2 + \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right].$$

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let A, B, C and a, b, c denote the angles and sides opposite these angles of the given triangle. Since the center of the inscribed circle is at the intersection of the bisectors of the angles of the triangle, the centers of the circles inscribed in the spandrils are on these bisectors. Let R be the radius of the inscribed circle and R_1, R_2, $\cdots R_n$ the radius of the circles whose centers are A_1, A_2, $\cdots A_n$, respectively.

From the triangle AOB we can easily show that

$$AO = \frac{c \sin B/2}{\sin (A+B)/2} \quad \text{and} \quad R = \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2};$$

and since

$$AA_1 = AO - A_1O = AO - R - R_1, \qquad AA_1 = \frac{R_1}{\sin A/2}.$$

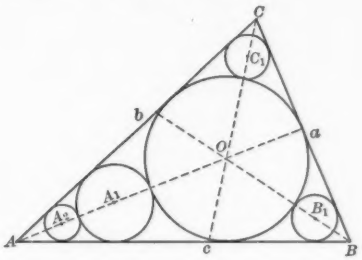
Hence, we have

$$\frac{R_1}{\sin A/2} = \frac{c \sin B/2}{\sin (A+B)/2} - \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2} - R_1.$$

Solving for R_1 we get

$$R_1 = \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2} \cdot \frac{1 - \sin A/2}{1 + \sin A/2}.$$

In the same way we can prove from $AA_2 = AO - A_2O$ that



$$R_2 = \frac{c \sin A/2 \sin B/2}{\sin (A+B)/2} \cdot \left(\frac{1 - \sin A/2}{1 + \sin A/2}\right)^2,$$

and finally, for circles on OA, $R_n = R(F_a)^n$; and similarly for circles on OB, $R_n = R(F_b)^n$; for circles on OC, $R_n = R(F_c)^n$, where

$$F_a = \frac{1 - \sin A/2}{1 + \sin A/2}, \qquad F_b = \frac{1 - \sin B/2}{1 + \sin B/2}, \qquad F_c = \frac{1 - \sin C/2}{1 + \sin C/2}.$$

The sum Σ of the areas of all these circles will then be

$$\Sigma = \pi R^2 + \pi R^2(F_a)^2 + \pi R^2(F_a)^4 + \pi R^2(F_a)^6 + \cdots \pi R^2(F_a)^{2n}$$
$$+ \pi R^2(F_b)^2 + \pi R^2(F_b)^4 + \cdots \pi R^2(F_b)^{2n} + \pi R^2(F_c)^2 + \pi R^2(F_c)^4 + \cdots \pi R^2(F_c)^{2n};$$

or,

$$\Sigma = \pi R^2[1 + (F_a)^2 + (F_a)^4 + \cdots (F_a)^{2n} + 1 + (F_b)^2 + (F_b)^4$$
$$+ \cdots (F_b)^{2n} + 1 + (F_c)^2 + (F_c)^4 + \cdots (F_c)^{2n} - 2].$$

Adding the terms of each of these geometric series by the usual formula, we have

$$\Sigma = \pi R^2 \left[\frac{(F_a)^{2n} - 1}{(F_a)^2 - 1} + \frac{(F_b)^{2n} - 1}{(F_b)^2 - 1} + \frac{(F_c)^{2n} - 1}{(F_c)^2 - 1} - 2\right].$$

If n approaches infinity, we have

$$\Sigma = \pi R^2 \left[\frac{1}{1 - (F_a)^2} + \frac{1}{1 - (F_b)^2} + \frac{1}{1 - (F_c)^2} - 2\right].$$

Replacing the values of F_a, F_b, and F_c we get

$$\Sigma = \frac{\pi R^2}{4}\left[\frac{(1 + \sin A/2)^2}{\sin A/2} + \frac{(1 + \sin B/2)^2}{\sin B/2} + \frac{(1 + \sin C/2)^2}{\sin C/2} - 2\right].$$

Let Δ be the area of the given triangle, then $\Delta = \frac{1}{2}(a+b+c)R$ or $R = \Delta S$ where $S = \frac{1}{2}(a+b+c)$.

Substituting the value of R in Σ, expanding the binomials, and reducing we finally get,

$$\Sigma = \frac{\pi}{4} \cdot \frac{\Delta^2}{S^2}\left[\frac{1}{\sin A/2} + \frac{1}{\sin B/2} + \frac{1}{\sin C/2} - 2 + \sin A/2 + \sin B/2 + \sin C/2\right].$$

Also solved by F. R. MORRIS, HORACE OLSON, G. W. HARTWELL, J. W. CLAWSON, PAUL CAPRON, and J. W. CROMWELL.

484. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that when spheres of uniform size are packed in the closest possible manner there is, in the interior of the mass, about 26 per cent. of voids.

SOLUTION BY LAENAS G. WELD, Pullman, Ills.

The given space may be divided into (equal) rhombic dodecahedrons. Spheres inscribed to these will be packed in the closest possible manner, since each has contact with twelve equal spheres. Now, the rhombic dodecahedron may be conceived as follows: Place a cube and a regular octahedron in such relation that each of the twelve edges of either figure is bisected at right angles by an edge of the other; then join the extremities of the edges so related, thus forming twelve rhombs, which are the faces of the figure in question. Each rhomb has for one of its diagonals an edge of the cube and for the other the corresponding edge of the octahedron. Taking the edge of the octahedron as 1, that of the cube related to it as above is equal to $\frac{1}{2}\sqrt{2}$. Moreover the radius of the inscribed sphere is equal to $\frac{1}{2}$, which is also the altitude of each of the twelve rhombic pyramids into which the dodecahedron may be resolved. The area of the base of each of these pyramids is one half the product of its diagonals, which is equal to $\frac{1}{4}\sqrt{2}$; whence the volume of the dodecahedron is

$$D = 12(\tfrac{1}{3} \cdot \tfrac{1}{2} \cdot \tfrac{1}{4}\sqrt{2}) = \tfrac{1}{2}\sqrt{2}.$$

The volume of the sphere is $S = \pi/6$.

The ratio S/D is equal to the ratio of the aggregate volume of the spheres to that of the space in which they are packed, since this space is completely filled by the circumscribing rhombic dodecahedrons. This ratio is 0.7405−. The voids, therefore, occupy 0.2595 + of the space, or about 26 per cent.

Also solved with slightly different results by PAUL CAPRON, G. PAASWELL, J. W. CLAWSON, and HERBERT N. CARLETON.

CALCULUS.

400. Proposed by H. S. UHLER, Yale University.

The axis of a prism whose right section is a regular polygon of apothem a and n sides passes through the center of a sphere of radius R. Show that, in general, the volume may be expressed by the formula:

$$V = \tfrac{4}{3}\pi R^3 + \tfrac{2}{3}a^2 n\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n}$$

$$+ \tfrac{1}{3}an(3R^2 - a^2)\sin^{-1}\left[\frac{2a\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n}}{R^2 - a^2}\right] - \tfrac{4}{3}nR^3\sin^{-1}\left[\frac{R\sin\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right].$$

Also discuss the special cases when $a = R\cos(\pi/n)$ and when $n = \infty$.

SOLUTION BY R. K. MORLEY, Worcester, Mass.

Using cylindrical coördinates

$$V = 4n\int_0^{\pi/n}\int_0^{a\sec\theta}\int_0^{\sqrt{R^2-\rho}} \rho dz d\rho d\theta = -\tfrac{2}{3}n\int_0^{\pi/n}(R^2 - a^2\sec^2\theta)^{3/2}d\theta + \tfrac{2}{3}n\int_0^{\pi/n}R^3 d\theta$$

$$= -\tfrac{2}{3}n\int_0^{\pi/n}(R^2 - a^2\sec^2\theta)(R^2 - a^2\sec^2\theta)^{1/2}d\theta + \tfrac{4}{3}\pi R^3.$$

For brevity, set $c^2 = R^2 - a^2$. Then

$$V = -\tfrac{1}{3}n\int_0^{\pi/n} \frac{R^3(R^2 - a^2\sec^2\theta)d\theta}{(R^2 - a^2\sec^2\theta)^{1/2}} + \tfrac{1}{3}na\int_0^{\pi/n} (c^2 - a^2\tan^2\theta)^{1/2}\cdot a\sec^2\theta d\theta + \tfrac{4}{3}\pi R^3,$$

$$= -\tfrac{1}{3}nR^3\int_0^{\pi/n} \frac{R\cos\theta\, d\theta}{(c^2 - R^2\sin^2\theta)^{1/2}} + \tfrac{1}{3}nR^2a\int_0^{\pi/n} \frac{a\sec^2\theta d\theta}{(c^2 - a^2\tan^2\theta)^{1/2}}$$

$$+ \tfrac{1}{3}na\left[\frac{a\tan\frac{\pi}{n}}{2}\sqrt{c^2 - a^2\tan^2\frac{\pi}{n}} + \frac{c^2}{2}\sin^{-1}\left(\frac{a\tan\frac{\pi}{n}}{c}\right)\right] + \tfrac{4}{3}\pi R^3,$$

$$= -\tfrac{1}{3}nR^3\sin^{-1}\left(\frac{R\sin\frac{\pi}{n}}{c}\right) + \tfrac{1}{3}nR^2a\sin^{-1}\left(\frac{a\tan\frac{\pi}{n}}{c}\right)$$

$$+ \tfrac{2}{3}na^2\tan\frac{\pi}{n}\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}} + \tfrac{2}{3}na(R^2 - a^2)\sin^{-1}\left(\frac{a\tan\frac{\pi}{n}}{c}\right) + \tfrac{4}{3}\pi R^3,$$

$$= \tfrac{4}{3}\pi R^3 + \tfrac{2}{3}a^2n\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}}\tan\frac{\pi}{n} - \tfrac{1}{3}nR^3\sin^{-1}\left[\frac{R\sin\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right]$$

$$+ \tfrac{2}{3}an(3R^2 - a^2)\sin^{-1}\left[\frac{a\tan\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right]. \quad (1)$$

This form for the result is more convenient than that required, but if desired this may be transformed into the required form by means of the relation $2A = \sin^{-1}(2\sin A\cos A)$ applied to the last term.

Special Case I. $a = R\cos(\pi/n)$. Then the polygonal cross-section of the prism is inscribed in a great circle of the sphere, and

$$V_1 = \text{volume of sphere} - n \text{ spherical segments of altitude } R - a$$

$$= \tfrac{4}{3}\pi R^3 - n\pi\left[R(R-a)^2 - \frac{(R-a)^3}{3}\right],$$

$$= \tfrac{4}{3}\pi R^3 - n\pi\left(\frac{2}{3}R^3 - R^2a + \frac{a^3}{3}\right).$$

Putting $a = R\cos(\pi/n)$ in (1) throughout except in the coefficient of the last term and reducing

$$V_1 = \tfrac{4}{3}\pi R^3 - \tfrac{2}{3}\pi nR^3 + \pi nR^2a - \tfrac{1}{3}\pi na^3,$$

$$= \tfrac{4}{3}\pi R^3 - n\pi\left(\frac{2}{3}R^3 - R^2a + \frac{a^3}{3}\right),$$

as before.

Special Case II. $n = \infty$. In this case there are some indeterminates to be evaluated. From the second term of (1)

$$\lim_{n=\infty} n\tan\frac{\pi}{n}\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}} = \lim_{n=\infty}\frac{\tan\frac{\pi}{n}}{\frac{1}{n}}\cdot\lim_{n=\infty}\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}}$$

$$= \lim_{n=\infty}\frac{\sec^2\frac{\pi}{n}\cdot\left(-\frac{\pi}{n^2}\right)}{-\frac{1}{n^2}}\cdot\sqrt{R^2 - a^2} = \pi\sqrt{R^2 - a^2}.$$

From the third term of (1),

$$\lim_{n=\infty}\left[\frac{\sin^{-1}\frac{R\sin\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}}{\frac{1}{n}}\right] = \lim_{n=\infty}\left[\frac{\frac{R\cos\frac{\pi}{n}\cdot\left(-\frac{\pi}{n^2}\right)}{(R^2 - a^2)^{1/2}\sqrt{1 - \frac{R^2}{R^2 - a^2}\sin^2\frac{\pi}{n}}}}{-\frac{1}{n^2}}\right] = \frac{R\pi}{(R^2 - a^2)^{1/2}}.$$

From the last term of (1) in a similar manner

$$\lim_{n=\infty} n \sin^{-1}\left[\frac{a \tan\frac{\pi}{n}}{(R^2-a^2)^{1/2}}\right] = \frac{\pi a}{(R^2-a^2)^{1/2}}.$$

Substituting these values in (1) and reducing

$$V_{II} = \lim_{n=\infty} V = \tfrac{1}{3}\pi[R^3 - (R^2-a^2)^{3/2}]$$

which is the result for the familiar problem of the volume cut from a sphere of radius R by circular cylinder of radius a when the center of the sphere is on the axis of the cylinder.

Also solved by PAUL CAPRON, A. W. SMITH and the PROPOSER.

401. Proposed by LAENAS G. WELD, Pullman, Ill.

Given a continuum of triangles whose sides are in arithmetical progression, the common difference being h: (a) The ratio of the mean value of all the triangles, the mean of whose three sides is not greater than μ, to the area of the triangle, the mean of whose three sides is equal to μ, is $(\mu + 2h)/3\mu$. Indicate the limiting values of this ratio and show that, when it is equal to 1/2, the triangle whose mean side is μ is right angled. (b) The ratio of the mean value of the areas of the circles inscribed in all these triangles, the mean of whose three sides is not greater than μ, to that of the circle inscribed in the triangle, the mean of whose three sides is equal to μ, has the limiting values 1/2 and 1/3. When the triangle whose mean side is μ is right angled, the ratio in question is 4/9. (c) Of the circles circumscribed about these triangles the minimum has the radius $2h$.

SOLUTION BY A. H. WILSON, Haverford College.

Let $x - h$, x, and $x + h$ be the three sides. Then the area is

$$[s(s-a)(s-b)(s-c)]^{1/2} = \tfrac{1}{4}[3x^2(x^2-4h^2)]^{1/2}.$$

The mean area of all triangles, the mean of whose three sides is not greater than μ, is

$$\frac{\sqrt{3}}{4(\mu-2h)}\int_{2h}^{\mu} x\sqrt{x^2-4h^2}\,dx = \frac{\sqrt{3}}{12}\sqrt{\mu^2-4h^2}(\mu+2h),$$

the least value of x being $2h$.

The area of the triangle for which $x = \mu$ is

$$\tfrac{1}{4}[3\mu^2(\mu^2-4h^2)]^{1/2} = \frac{\sqrt{3}}{4}\mu(\mu^2-4h^2)^{1/2}.$$

(a) The ratio of these two areas is $(\mu + 2h)/3\mu$. The limits of this ratio occur for the values $\mu = 2h$ and $\mu = \infty$, and are, respectively, 2/3 and 1/3. If the ratio is equal to 1/2, then $\mu = 4h$; and the sides $3h$, $4h$, and $5h$ are those of a right triangle.

(b) The area of the inscribed circle is

$$\pi r^2 = \pi(x^2-4h^2)/12. \quad (r = [(s-a)(s-b)(s-c)/s]^{1/2}).$$

The mean of such areas for triangles of the first class is

$$\frac{\pi}{12(\mu-2h)}\int_{2h}^{\mu}(x^2-4h^2)\,dx = \frac{\pi}{36}(\mu^2+2h\mu-8h^2).$$

The area of the circle of the second class is $\pi(\mu^2-4h^2)/12$; and the ratio of the two areas is $(\mu + 4h)/3(\mu + 2h)$. The limiting values of this ratio are 1/2 and 1/3, and for the value 4/9 the mean side is $4h$, and the triangle is right-angled.

(c) The area of the circumscribed triangle is $\pi R^2 = \pi(abc)^2/16s(s-a)(s-b)(s-c) = \pi(x^2-h^2)^2/3(x^2-4h^2)$. For a minimum, equate the derivative to 0; and there results for x the value $\sqrt{7}h$, giving a minimum. For this value of x the radius of the circumscribed circle is $2h$.

Also solved by ELIJAH SWIFT.

402. Proposed by C. N. SCHMALL, New York City.

If (x, y) be a double point on the curve $u \equiv f(x, y) = 0$, show that (1) the two branches of the curve will cut orthogonally if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

and (2), if this point be made the origin, then the equation of the tangents to the branches will be

$$(y'^2 - x'^2)\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x \partial y} = 0$$

where (x', y') are the current coördinates of points on the tangents.

SOLUTION BY C. E. DIMICK, New London, Connecticut.

If (x, y) be a double point of the curve $u \equiv f(x, y) = 0$, $\partial u/\partial x = \partial u/\partial y = 0$ and the two values of dy/dx at the double point must satisfy the quadratic

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

(Todhunter's Diff. Calc., pages 319, 320.)

If the two tangents are perpendicular, the product of their slopes is -1, and since the product of the roots of the quadratic $a + bx + cx^2 = 0$ is a/c we have

$$\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = -1 \qquad \text{or} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The equations of the two tangents will be $y - y_1 = l_1(x - x_1)$, and $y - y_1 = l_2(x - x_1)$, where l_1 and l_2 are the roots of the quadratic given above. Transposing and multiplying, the equation of the two tangents taken together will be

$$(y - y_1)^2 - (x - x_1)(y - y_1)(l_1 + l_2) + l_1 l_2 (x - x_1)^2 = 0,$$

which becomes on substituting the values of the sum and product of the roots of the quadratic

$$(y - y_1)^2 + 2(x - x_1)(y - y_1)\frac{\partial^2 u}{\partial x \partial y}\Big/\frac{\partial^2 u}{\partial y^2} + (x - x_1)^2\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = 0,$$

which, upon transforming to (x_1, y_1) as origin, becomes

$$y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial x^2} = 0.$$

But $\partial^2 u/\partial x^2 = -\partial^2 u/\partial y^2$ as the tangents are perpendicular.

Hence the equation reduces to

$$(y^2 - x^2)\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0,$$

the minus sign as given in the problem being incorrect.

Also solved by I. A. BARNETT, C. K. ROBBINS, GRACE M. BAREIS, J. A. BULLARD, F. M. MORGAN, and H. L. AGARD.

MECHANICS.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.

Solution by H. S. Uhler, Yale University.

Let f denote the force exerted by the solid sphere of mass m_1 on the shell of radius of gyration k and of mass m. Also, let (x_1, y_1) and (x, R) be the coordinates of the centers $(C_1$ and $C)$ of the solid and hollow bodies, respectively. f, x, x_1, and y_1 will be functions of the time t. By using the symbols k, m, and m_1 the equations given below apply to two cases in addition to the one proposed, namely, (i) a sphere within a cylindrical shell, and (ii) a cylinder inside a hollow cylinder. Moreover, in all three cases, the bodies are only required to have their centers of mass coincide with their geometric centers, so that the inner body may be hollow and all the bodies may be non-homogeneous radially, that is, the bodies may be built up of similar concentric or coaxial shells of different densities for the several layers. Neither does the density have to be a continuous function of the radius. In the given case $k^2 = 2(R^5 - r^5)/[5(R^3 - r^3)]$, $m = \frac{4}{3}\pi\delta(R^3 - r^3)$, and $m_1 = \frac{4}{3}\pi\delta r_1^3$, where δ symbolizes the constant density of the material involved.

Consider the outer body and take moments around the instantaneous axis through A. Then

$$m(k^2 + R^2)\frac{d^2\phi}{dt^2} = mgR\sin\alpha + fR\sin\theta,$$

where θ is the angle which the line $\overline{C_1C}$ makes at any time t with the fixed direction $\overline{AC}$ or $\overline{OY}$. The geometric meaning of ϕ is obvious from its defining equation $R\phi = x$ (no slipping of the body on the inclined plane). Then

$$\frac{d^2\phi}{dt^2} = \frac{1}{R}\cdot\frac{d^2x}{dt^2}$$

so that, writing s for $1 + k^2/R^2$,

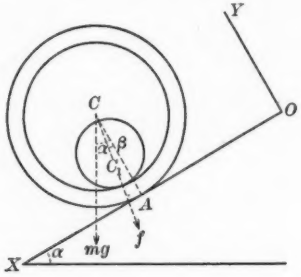
$$ms\frac{d^2x}{dt^2} = mg\sin\alpha + f\sin\theta. \tag{1}$$

The forces acting parallel to $\overline{OX}$ on the inner body fulfil the equation

$$m_1\frac{d^2x_1}{dt^2} = m_1g\sin\alpha - f\sin\theta. \tag{2}$$

The forces parallel to $\overline{OY}$ on this body satisfy the relation

$$m_1\frac{d^2y_1}{dt^2} = -m_1g\cos\alpha + f\cos\theta. \tag{3}$$



From the diagram, we see that

$$x_1 - x = \rho\sin\theta, \tag{4}$$

and

$$R - y_1 = \rho\cos\theta, \tag{5}$$

where $\rho \equiv r - r_1$, $(r > r_1)$.

Of the several differential equations that can be derived from the preceding five relations, the one connecting θ and t is the simplest.

Differentiating (4) and (5) twice each with respect to t we obtain, respectively,

$$\frac{d^2x_1}{dt^2} - \frac{d^2x}{dt^2} = \rho\cos\theta\frac{d^2\theta}{dt^2} - \rho\sin\theta\left(\frac{d\theta}{dt}\right)^2 \tag{4'}$$

and

$$\frac{d^2y_1}{dt^2} = \rho \sin\theta \frac{\theta d^2\theta}{dt^2} + \rho \cos\theta \left(\frac{d\theta}{dt}\right)^2. \tag{5'}$$

Multiplying equations (1), (2), (3), (4'), and (5') by $m_1 \cos\theta$, $-ms\cos\theta$, $-(ms+m_1)\sin\theta$, $mm_1 s\cos\theta$, and $m_1(ms+m_1)\sin\theta$, in the order named, and then adding, we get

$$\rho(ms + m_1\sin^2\theta)\frac{d^2\theta}{dt^2} + m_1\rho\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2 + g(ms+m_1)\cos\alpha\sin\theta - \frac{mgk^2}{R^2}\sin\alpha\cos\theta = 0. \tag{6}$$

Equation (6) may be reduced to the first order by observing that

$$2m_1\sin\theta\cos\theta = \frac{d}{d\theta}(ms + m_1\sin^2\theta), \qquad \text{and} \qquad \frac{d^2\theta}{dt^2} = \frac{d\theta}{dt}\cdot\frac{d(d\theta/dt)}{d\theta}.$$

Accordingly

$$\rho(ms + m_1\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2 - 2g(ms+m_1)\cos\alpha\cos\theta - \frac{2mgk^2}{R^2}\sin\alpha\sin\theta + c' = 0. \tag{7}$$

When $t = 0$, let $\theta = \theta_0$ and $d\theta/dt = \omega_0$, giving

$$c' = 2g(ms+m_1)\cos\alpha\cos\theta_0 + (2mgk^2/R^2)\sin\alpha\sin\theta_0 - \rho(ms + m_1\sin^2\theta_0)\omega_0^2.$$

Consequently,

$$t + f(\theta_0) = \int \frac{d\theta\sqrt{\rho(ms + m_1\sin^2\theta)}}{\sqrt{\rho(ms + m_1\sin^2\theta_0)\omega_0^2 + 2g(ms+m_1)\cos\alpha(\cos\theta - \cos\theta_0) - \frac{2mgk^2}{R^2}\sin\alpha(\sin\theta_0 - \sin\theta)}}. \tag{8}$$

Since the analysis has been explicitly reduced to the evaluation of an indefinite integral the problem may be considered as formally solved. For, equation (8) theoretically gives θ as a function of t so that (5) then expresses the dependence of y_1 (and so also of dy_1/dt and d^2y_1/dt^2) upon t. Or, we may substitute directly in (5') to get $d^2y_1/dt^2 = F(\theta) = \Phi(t)$. By suitably combining (1), (2), and (4') it is ideally possible to exhibit x, x_1, dx/dt, dx_1/dt, d^2x/dt^2, and d^2x_1/dt^2 as explicit functions of t. However, since the writer does not know how to evaluate the integral in (8), save as a series development, he cannot profitably continue the general solution. Of course, the periodic nature and other properties of the function can be readily demonstrated but nothing is thereby added to what we know in advance from purely dynamical considerations.

Nevertheless, in the course of the investigation, we have noticed certain general and special facts which seem to be interesting. In the first place, elimination of f from (2) and (3) gives

$$\cos\theta\frac{d^2x_1}{dt^2} + \sin\theta\frac{d^2y_1}{dt^2} = g\sin(\alpha - \theta). \tag{9}$$

This equation shows that the sum of the components normal to $\overline{CC_1}$ of d^2x_1/dt^2 and d^2y_1/dt^2 is equal to the component in the same direction of the acceleration due to gravity, that is, the total linear acceleration of m_1 parallel to the common tangent of the inner and outer bodies is equal to the component of g in the same direction. This is a natural consequence of the perfect smoothness of the surfaces of contact. Relation (9) also indicates explicitly how d^2x_1/dt^2 can be obtained as a function of t as soon as θ and y_1 have been evaluated in terms of the time. Multiplying equations (4') and (5') in order by $\cos\theta$ and $\sin\theta$, and then subtracting the sum from (9) we find

$$\rho\frac{d^2\theta}{dt^2} + \cos\theta\frac{d^2x}{dt^2} = g\sin(\alpha - \theta). \tag{10}$$

This equation admits of the same physical interpretation as (9), relative motion of the bodies now being involved, and shows how to get d^2x/dt^2 directly as a function of the time.

As a second case, the result of adding equations (1) and (2) is

$$ms\frac{d^2x}{dt^2} + m_1\frac{d^2x_1}{dt^2} = g(m+m_1)\sin\alpha. \tag{11}$$

If we take a point (ξ, η) which divides the line joining $C_1(x_1, y_1)$ to $C(x, R)$ internally in the ratio $ms : m_1$, then $\xi = (msx + m_1x_1)/(ms + m_1)$ so that (11) becomes

$$\frac{d^2\xi}{dt^2} = \frac{(m+m_1)g\sin\alpha}{ms + m_1}. \tag{12}$$

Therefore, this point, which is definitely associated with the moving system, has the property of constant acceleration parallel to the inclined plane. Equation (12) can be integrated at once giving $d\xi/dt$ and ξ as explicit functions of the time, quite independently of θ. On the other hand, $\eta = (msR + m_1y_1)/(ms + m_1)$; hence,

$$\frac{d^2\eta}{dt^2} = \frac{m_1}{ms + m_1} \cdot \frac{d^2y_1}{dt^2},$$

which is, in general, a function of θ.

An important special case of the motion of the system of bodies is brought out by (6). This equation is satisfied for all time if θ maintains the constant value θ' given by

$$\tan\theta' = \frac{mk^2}{mk^2 + (m + m_1)R^2} \cdot \tan\alpha; \tag{13}$$

for then $d^2\theta/dt^2 = d\theta/dt = 0$. Relation (13) shows that $\alpha > \theta' > 0$, as in the diagram. Under these conditions (4) and (5) give, respectively, $dx_1/dt = dx/dt$ and $dy_1/dt = 0$. Consequently, if we start with the centers of the inner and outer bodies at the respective points $(x_0 + \rho\sin\theta', R - \rho\cos\theta')$ and (x_0, R), and impart equal linear velocities parallel to the incline, the inner body will not oscillate relative to its constraining wall but will maintain the constant angular position θ' while the shell slips under it. The equations of motion of the bodies are now easily shown to be

$$(ms + m_1)\frac{d^2x}{dt^2} = (ms + m_1)\frac{d^2x_1}{dt^2} = g(m + m_1)\sin\alpha,$$

$$(ms + m_1)\frac{dx}{dt} = (ms + m_1)\frac{dx_1}{dt} = g(m + m_1)\sin\alpha \cdot t + (ms + m_1)v_0,$$

$$(ms + m_1)x = \tfrac{1}{2}g(m + m_1)\sin\alpha \cdot t^2 + (ms + m_1)(v_0t + x_0),$$

$$(ms + m_1)x_1 = \tfrac{1}{2}g(m + m_1)\sin\alpha \cdot t^2 + (ms + m_1)(v_0t + x_0 + \rho\sin\theta').$$

Finally, if we introduce a new variable angle λ, defined by the equation $\lambda = \theta - \theta'$, and make use of (13) we can change the expression under the radical in the denominator of the integral in (8) to the form

$$\rho(ms + m_1\sin^2\theta_0)\omega_0^2 + 4g(ms + m_1)\cos\alpha\sec\theta'\sin\tfrac{1}{2}[(\theta' - \theta_0) + \lambda]\sin\tfrac{1}{2}[(\theta' - \theta_0) - \lambda],$$

which assumes the same value when λ is assigned values which are numerically equal but of opposite sign. Unfortunately for the analysis, the numerator of the integrand does not possess this kind of symmetry.

No other correct solution of this problem was received. Should any one integrate equation (8) we shall be glad to publish the result.—EDITORS.

NUMBER THEORY.

223. (October, 1914) Proposed by T. E. MASON, Purdue University.

Show that

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}}$$

is an integer, r, s, and t being positive integers. Generalize to the case of n integers, r, s, t, u, $\cdots$.

[Carmichael's *Theory of Numbers*, page 28.]

SOLUTION BY FRANK IRWIN, University of California.

Suppose we have rst objects, and let us divide them into t classes of rs objects each, then each class into s sub-classes of r objects each, and let us call each such classification, without any reference to order, a "classification" *par excellence*. We assert that the total number of such classifications is

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}},$$

which expression is, consequently, an integer.

For, let us set up an order among the classes, among the sub-classes in each class, and among the objects in each sub-class. Then from any given classification we may, by permutations of the objects which make changes in the order merely, but do not change the constituents of the classes and sub-classes—which, then, leave the classification as such the same—we may, by such operations, obtain various permutations of the totality of the rst objects. First, without changing the order of the classes or sub-classes, we may permute among themselves the objects in each sub-class. For any one sub-class this may be done in $r!$ ways, and, therefore, for all the st sub-classes in $(r!)^{st}$ ways. Next, we may permute among themselves the sub-classes in each class. For any one class this may be done in $s!$ ways; for all the t classes, then, in $(s!)^t$ ways. Finally, we may permute the classes in $t!$ ways.

Combining these various kinds of permutations, we see that each classification gives rise to $t!(s!)^t(r!)^{st}$ permutations of the rst objects. Since each of these latter permutations may be derived, by the process explained, from some classification or other, and two different classifications cannot give rise to the same permutation, it follows that the number of permutations of the rst objects is $t!(s!)^t(r!)^{st}$ times the number of classifications; that is, since there are $(rst)!$ permutations there are, as was to be shown, $(rst)!/t!(s!)^t(r!)^{st}$ classifications.

The argument may evidently be generalized to show (with a change in the notation) that

$$\frac{(r_1r_2\cdots r_n)!}{r_1!(r_2!)^{r_1}(r_3!)^{r_1r_2}\cdots(r_n!)^{r_1r_2\cdots r_{n-1}}}$$

is an integer.

228. Proposed by HERMON C. KATANIK,* Indianapolis, Ind.

Deduce a formula for the difference between any two squares, and thus show that (1) The difference between any two consecutive squares is of the form $2n + 1$; (2) The difference between any two squares is even or odd according to whether they are separated by an odd or even number of squares; (3) The differences of the squares of the consecutive terms of any arithmetic progression form another arithmetic progression.

SOLUTION BY WALTER C. EELLS, Whitman College.

Let T_i be the ith term. Then $T_n = n^2$, $T_{n+k} = (n + k)^2$.

$$T_{n+k} - T_n = (n + k)^2 - n^2 = (n + k - n)(n + k + n) = k(2n + k).$$

(1) If $k = 1$, $T_{n+1} - T_n = (2n + 1)$.

(2) When separated by an odd number of squares $T_{n+2k'} - T_n = 2k'(2n + 2k')$, which is even. When separated by an even number of squares $T_{n+2k'+1} - T_n = (2k' + 1)(2n + 2k' + 1)$, which is odd.

(3) Since k is constant for any given arithmetic series, $T_{n+mk} - T_{n+(m-1)k}$ is constant and is the constant difference for another arithmetic series.

Also solved by HORACE OLSON and HERBERT N. CARLETON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence, Kans.

A number of questions published in this department have been standing for some time without having been answered. We are re-publishing these in full this month in the hope that some of our readers may thereby be stimulated to send in suitable replies.

12. In view of the notation used by Professor Slobin in his "Note on Certain Algebraic Equations" published in the MONTHLY for April, 1914, pages 113–115, a discussion would be desirable as to the best notation for complex roots in general, and in particular for eliminating the conspicuous ambiguities introduced by the notation above cited.

15. We are in receipt of the following communication from Mr. W. E. Heal, of Washington, D. C.: "In the Proceedings of the Royal Society of Edinburgh, Vol. VII, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

* Deceased since proposing this problem.

"'If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"'This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube.'

"The writer has failed to see how this 'easy procf' follows and has been unable to find the question discussed or even mentioned in Tait's collected works. Can some reader of the MONTHLY supply the missing link or links?"

20. Some of our readers would like to have a simple account, without proofs, of just what has been accomplished toward the proof of the theorem that the equation $x^n + y^n = z^n$ is impossible in integers when $n > 2$.

21. For the diophantine equation

$$x^3 - y^3 = 17$$

there are known the following solutions:

$$x = \quad 3, \quad 4, 5, 9, 23, 282, 375, 378661,$$

$$y = -2, -1, 2, 4, \quad 5, \quad 43, \quad 52, \quad 5234.$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

26. Why should not the nomenclature of mathematics be made uniform? For example, why call a circle a *portion of a plane* in elementary geometry and a *curve* in analytic geometry? Why call a sphere a *ball* at one time and a *surface* at another time? And so on through all the configurations of two- and three-dimensional geometry.

28. Is it possible to obtain $\int \cos\theta^2 d\theta$ without expanding $\cos\theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

30. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

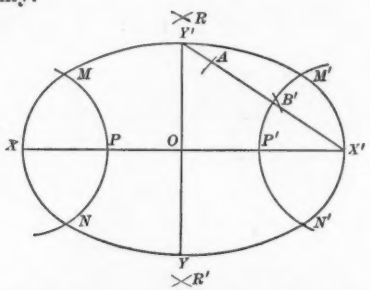
NEW QUESTION.

33. Under what conditions or to what extent is Mr. Iwerson's construction, given below, a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

AN APPROXIMATE CONSTRUCTION FOR AN ELLIPSE.

By RICHARD IWERSON, Everett, Washington.

Given the axes of an ellipse, to construct approximately the curve by use of ruler and compasses only.



Draw XX', the major axis, and YY', the minor axis, perpendicular to and bisecting each other at O. Draw the line $X'Y'$. With O as center and Ox as radius describe an arc cutting $X'Y'$ at B. With Oy as radius and X' as center

describe an arc cutting $X'Y'$ at A. With AB as radius and X as center describe an arc cutting XO at P and extending on each side of XO toward Y and Y'. With the same radius (AB) and P as center draw an arc through X cutting the the arc just drawn in the points M and N. Construct in similar manner the arc $M'N'$ through X'. With NN' as radius and N and N' as centers describe arcs toward Y' cutting each other, obviously, in some point R on OY' (produced, if necessary). With R as center and RN as radius draw the arc NN'. In similar manner draw the arc MM'. The curve $XMY'M'X'N'YNX$ is the required approximation.

Note.—The above construction obviously can not be a true ellipse, except for $a = b$, since the curvature on the arc NYN' is constant while the curvature of the ellipse is variable $(= a^4b^4/(b^4x^2 + a^4y^2)^{3/2}$. This suggests that the construction is a close approximation only in case the difference $a - b$ is small. There are, however, important applications of ellipses, such, for example, as the paths of the planets, in which the difference $a - b$ is small. In such cases Mr. Iwerson's construction might give valuable results if there were some definite criterion (possibly some function of the difference $a - b$ or of the eccentricity of the ellipse) which would readily and accurately measure the degree of approximation. Question 33 above is asked in the hope that some of our readers may be able to furnish such a criterion or suggest useful applications.—U. G. M.

DISCUSSIONS.

Relating to the Number of Terms between Two Given Terms of an Ordered Polynomial.

By O. E. Glenn, University of Pennsylvania.

A complete homogeneous polynomial P of order m in n letters $x_1, x_2, \cdots, x_n$ may be said to have its terms arranged in normal order when the exponents in any two terms of P, as

$$t = C_{k_1 k_2 \cdots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n},\ t' = C_{l_1 l_2 \cdots l_n} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n},$$

where t comes before t', satisfy the condition that the first non-vanishing difference of the set

$$k_n - l_n,\ k_{n-1} - l_{n-1},\ \cdots,\ k_p - l_p,\ \cdots,\ (k_1 - l_1),$$

is negative. We wish to prove that the number of terms of P between t and t', including t' but not including t, is given by the formula

$$(2)\qquad N\binom{k_1, \cdots, k_n}{l_1, \cdots, l_n} = \left\{ \begin{array}{l} \binom{l_1}{0} + \binom{l_1+1}{0} + \cdots + \binom{k_1-1}{0} \\ + \binom{l_1+l_2+1}{1} + \binom{l_1+l_2+2}{1} + \cdots + \binom{k_1+k_2}{1} \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \binom{l_1+\cdots+l_{n-2}+n-3}{n-3} + \binom{l_1+\cdots+l_{n-2}+n-2}{n-3} \\ \qquad\qquad + \cdots + \binom{k_1+\cdots+k_{n-2}+n-4}{n-3} \\ + \binom{l_1+\cdots+l_{n-1}+n-2}{n-2} + \binom{l_1+\cdots+l_{n-1}+n-1}{n-2} \\ \qquad\qquad + \cdots + \binom{k_1+\cdots+k_{n-1}+n-3}{n-2} \end{array} \right\},$$

where the numbers $\binom{a}{b}$ are ordinary binomial numbers, and are defined to be zero when b is negative.

All terms in this formula have plus signs provided

$$l_1 + \cdots + l_s < k_1 + \cdots + k_s (s = 1, \cdots, n-1).$$

If, for any of these values of s, $l_1 + \cdots + l_s = k_1 + \cdots + k_s$, the line number s, viz.,

$$\binom{l_1 + \cdots + l_s + s - 1}{s-1} + \binom{l_1 + \cdots + l_s + s}{s-1} + \cdots + \binom{k_1 + \cdots + k_s + s - 2}{s-1}$$

disappears from the formula. But if $l_1 + \cdots + l_s > k_1 + \cdots + k_s$ for a value of s, then line number s is to be replaced by, or is to be interpreted as signifying

$$-\binom{k_1 + \cdots + k_s + s - 1}{s-1} - \binom{k_1 + \cdots + k_s + s}{s-1} - \cdots - \binom{l_1 + \cdots + l_s + s - 2}{s-1}.$$

We have $k_1 + k_2 + \cdots + k_n = l_1 + l_2 + \cdots + l_n = m$, and P is ordered according to ascending powers of x_n, the coefficients of the various powers of x_n being polynomials in $n-1$ letters $x_1, \cdots, x_{n-1}$, *i. e.*, in a common terminology, are $(n-1)$-ary forms. Beginning to enumerate terms from the lower (right-hand) end of P we observe that $N\binom{k_1, k_2, \cdots, k_n}{0, 0, \cdots, m}$ equals 1 plus the number of terms in an $(n-1)$-ary linear form, plus the number of terms in an $(n-1)$-ary quadratic form, plus the number in a cubic, plus and so on to and including the number of terms in an $(n-1)$-ary $(m - k_n - 1)$-ic form; plus 1 plus the number of terms in an $(n-2)$-ary linear form, plus the number in an $(n-2)$-ary quadratic and so on to and including the number in an $(n-2)$-ary $(m - k_n - k_{n-1} - 1)$-ic form, and so forth. This double entry summation extends to include the (last) row 1 plus the number of terms in a linear unary form, plus the number in a quadratic unary form, plus and so on to and including the number in a unary $(m - k_n - k_{n-1} - \cdots - k_2 - 1)$-ic form. Hence

$$(3)\quad N\binom{k_1, \cdots, k_n}{0, \cdots, m} = \left\{ \begin{array}{l} \binom{0}{0} + \binom{1}{0} + \cdots + \binom{k_1 - 1}{0} \\ + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{k_1 + k_2}{1} \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \binom{n-3}{n-3} + \binom{n-2}{n-3} + \cdots + \binom{k_1 + \cdots + k_{n-2} + n - 4}{n-3} \\ + \binom{n-2}{n-2} + \binom{n-1}{n-2} + \cdots + \binom{k_1 + \cdots + k_{n-1} + n - 3}{n-2} \end{array} \right\}.$$

This serving as a general formula we have immediately

$$N\binom{k_1, \cdots, k_n}{0, \cdots, m} - N\binom{l_1, \cdots, l_n}{0, \cdots, m} = N\binom{k_1, \cdots, k_n}{l_1, \cdots, l_n},$$

equal to formula (2) above.

For illustration, a complete quaternary cubic in normal order is the following:

$$x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$$
$$+ x_1x_3^2 + x_2x_3^2 + x_3^3 + x_1^2x_4 + x_1x_2x_4 + x_2^2x_4 + x_1x_3x_4$$
$$+ x_2x_3x_4 + x_3^2x_4 + x_1x_4^2 + x_2x_4^2 + x_3x_4^2 + x_4^3.$$

The number of terms between $x_1^2x_3$ and $x_3x_4^2$ is

$$N\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} = \left\{ \begin{array}{c} \binom{0}{0} + \binom{1}{0} \\ + \binom{1}{1} + \binom{2}{1} \\ + \binom{3}{2} + \binom{4}{2} \end{array} \right\} = 14.$$

The total number of terms in a complete ternary form of order m is

$$1 + N\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & m \end{pmatrix} = \left\{ \begin{array}{c} \binom{0}{0} + \binom{1}{0} + \cdots + \binom{m-1}{0} \\ + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{m}{1} \end{array} \right\} + 1 = \tfrac{1}{2}(m+1)(m+2).$$

EXERCISE. Show that the number of terms between $x_1^3x_2^2x_3^2x_4^{m-7}$ and $x_1^2x_2x_3^2x_4^{m-6}$ in a complete ordered quaternary polynomial of order m ($m > 6$) is 59.

EXERCISE. Show that

$$N\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \end{pmatrix} = 31.$$

NOTES AND NEWS.

SEND COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

Dr. OTTO DUNKEL, formerly instructor at the University of Missouri, has been appointed assistant professor of mathematics at Washington University, St. Louis, Mo.

"Differential equations and implicit functions in infinitely many variables" is the title of a paper by Dr. W. L. HART of Harvard University, abstract of which appeared in the *Proceedings* of the National Academy of Sciences for June, 1916.

Mr. EDWARD B. ESCOTT is now connected with the auditing department of the Kansas City Life Insurance Company, Kansas City, Mo.

Mr. H. C. CLEVENGER, of Urbana, Ill., has accepted an instructorship in mathematics at the University of Minnesota.

Dr. R. L. BÖRGER, associate in mathematics at the University of Illinois, goes to Ohio University at Athens as professor of mathematics.

Dean THOMAS F. HOLGATE, professor of mathematics at Northwestern University, has been chosen by the trustees *ad interim* president of the University.

The *Washington University Studies*, January, 1916, contains a paper by Professor W. H. ROEVER on "Some recent work on the problem of the deviations of freely falling bodies." This paper is a sixteen-page résumé of the publications of ROEVER, CAJORI, HALL, WOODWARD, MOULTON, and others, which have appeared in *Science*, *Bulletin and Transactions of the American Mathematical Society*, the *Astronomical Journal*, *Physical Review* and other periodicals.

An expository volume on integral equations appeared in July, 1916, with the title *Elementi della Teoria delle Equazioni Integrali Lineari* by G. VIVANTI. The book appears in the extensive collection called, *Manuali Hoepli*, and contains about four hundred pages. In the preface the author refers to the present European war and to the need of scientific emancipation of Italy. The present volume is a contribution in that direction.

The Philosophical Transactions of the Royal Society of London, Vol. 216, 1916, has a paper by K. PEARSON, the noted biometrician, on "Mathematical contributions to the theory of evolution."

The July, 1916, number of *The Mathematical Gazette* contains the concluding article by E. H. NEVILLE, on "So-called cases of failure in the solution of linear differential equations" referred to in the October issue of the MONTHLY. The introductory part of this discussion appeared in the May issue of the *Gazette*.

The first number of Vol. 18 of *The Annals of Mathematics* has appeared containing the following papers: "On a surface of lowest degree passing through a given curve in space," by T. HYASHI; "A practical method of determining elementary divisors," by H. T. BURGESS; "Conjugate systems with equal point invariants," by L. P. EISENHART; "On the derivative of a function at a point," by J. F. RITT; "An existence theorem for the solution of a type of real mixed difference equations," by A. A. BENNETT; "Double elliptic geometry in terms of point and order alone," by J. R. KLINE; "An application of a group of order sixteen to a configuration of an elliptic cubic," by A. EMICH.

Parts II and III, of Vol. 40, *Rendiconti del Circolo Matematico di Palermo*, contain fourteen research papers among which are the following in English: "A new theorem in analytic conics," by A. HAWKESWORTH; "The meaning of Plücker's equations for a real curve," by J. L. COOLIDGE; "On Stieltjes's integral," by T. HAYASHI; "Plane nets periodic of period three under the Laplacian transformation," by J. O. HASSLER.

Miss ELEANORA HARRIS, head of the department of mathematics at the Hutchinson, Kan., high school, and secretary of the Kansas Association of Mathematics Teachers, has been markedly successful in conducting a mathematics club for the advanced students in the high school. They have discussed

the practical uses of graphs, of the slide rule, and of determinants of the second and third orders. They have studied the history of our numerals, of logarithms, and of the Pythagorean theorem. And they have debated the question whether one year of algebra and one year of geometry should not be required of all students for graduation from the high school.

The fourth annual meeting of the Mathematics Section of the California High School Teachers' Association was held in July at the University of California under the chairmanship of Professor HENRY W. STAGER, Fresno Junior College. The following program was presented: "Non-technical discussion of some questions of general interest in geometry," by Professor E. J. WILCZYNSKI, University of Chicago; "Mathematics as an applied subject," by Dr. E. R. SNYDER, California Commissioner of industrial and vocational education; "Minimum requirements in mathematics," by Professor HENRY W. STAGER, Fresno Junior College; "Some of the difficulties a mathematics teacher of to-day must face," by FRANK R. MORRIS, Glendale High School. The discussions centered largely upon the present attacks upon the teaching of mathematics in the secondary schools by the leaders of the vocational movement. It was urged that united effort be made to meet these attacks and that all teachers of mathematics as individuals and in organizations should continue to defend the cultural value of mathematics.

In order to make the work of the Section more continuous and more effective, a standing committee on policy, selected from all parts of the state, was authorized. An educational survey of the teaching of mathematics in California was also authorized, the survey to be undertaken by the committee on policy in coöperation with the State Commissioner for secondary schools. The official reading course for the past year as printed in the MONTHLY for March, 1916, pp. 77–78, was adopted for the year 1916–1917 without change. Professor C. A. NOBLE, University of California, was elected chairman of the Section, and Mr. G. E. MERCER, Palo Alto High School, secretary for the ensuing year.

The Association of Mathematics Teachers of New Jersey is the second, so far as we know, the California association being the first, to organize with the distinct purpose of giving to high school teachers the "upward look"; or, to use the words of one of the organizers of the New Jersey society, to choose as the field of activity what may be called "graduate work in high school mathematics." To this end, each program has papers emphasizing the "graduate," or in other words, the "collegiate" point of view. For instance, at one of their earlier meetings, Professor OSWALD VEBLEN, of Princeton University, gave a paper on "The affine geometry"; Professor RICHARD MORRIS, of Rutgers College, spoke on "The auxiliary angle"; Professor C. O. GUNTHER, of Stevens Institute of Technology, on "Trigonometry for the college student"; and Mr. HARRISON E. WEBB, of the Central High School of Newark, on "Geometric definitions of the trigonometric functions." Their subsequent programs have continued to include papers of this character.

Professor H. W. Tyler contributes the following biographical note:

"Professor WEBSTER WELLS, long a member of the Faculty of the Massachusetts Institute of Technology, died in a private hospital near Boston, May 23, 1916. Born in 1851, Professor Wells prepared for the Institute at the Allen School, West Newton, Massachusetts. He was graduated in civil engineering in 1873 and was then appointed instructor in mathematics. In 1880 he went to Germany for further study, spending two years in Leipsic, mainly in private mathematical work. Returning to the Institute in 1882, he continued teaching until his retirement on account of serious impairment of health in 1911. At the time of his return from Germany conditions were not yet favorable for the introduction of advanced or graduate courses, and Professor Wells accordingly devoted his unusual powers of lucid exposition to the preparation and publication of an extended series of text books in the whole field of elementary mathematics. These texts represent an enormous amount of careful labor in a field already so long cultivated. They are in general notable for the clear and skilful arrangement of traditional materials rather than for any attempt at innovations. Professor Wells was a member of the American Mathematical Society since 1895.

Outside of the field of mathematics, Professor Wells was especially interested in music and foreign travel; also early in life in mountaineering. He was for a short time Secretary of the Faculty, and Bursar of the Institute.

NOTES ON THE ASSOCIATION.

The first annual meeting of the Association will take place on Friday and Saturday, December 29, 30, 1916, at Columbia University, New York City. The selection of the place of meeting was determined by the fact that the American Association for the Advancement of Science meets in New York during holiday week, thus bringing together large numbers representing most of the national scientific societies of the country. In particular, the American Mathematical Society will hold its annual meeting at Columbia University on Wednesday and Thursday, December 27, 28, so that the juxtaposition of dates for the Association and the Society will make it convenient for mathematicians to attend both meetings.

The program of the annual meeting of the Association will be in charge of a committee consisting of Professor D. E. Smith of Columbia University, Chairman, Professor E. H. Moore of the University of Chicago, and Professor G. D. Olds of Amherst College. A special sub-committee consisting of Professor J. N. Van der Vries of the University of Kansas, Chairman, Professor J. N. Hart of the University of Maine, and Professor Helen Merrill of Wellesley College, will arrange the program for the meeting of institutional delegates.

The committee on arrangements for the New York meeting consists of Professor T. S. Fiske of Columbia University, Chairman, Professor T. W. Edmondson of New York University, Professor Paul Saurel of the College of the City of New York, Professor Emma Requa of Hunter College, and Professor D. E. Smith of Teachers College. All these committees are authorized to extend

their membership as circumstances may dictate. They will issue preliminary reports as early as seems feasible. It is hoped that the December issue of the MONTHLY may contain a full outline of the programs and arrangements.

Another important committee of the Association has just been organized, namely, the Library Committee which was authorized last spring and consists of the following persons: Professor W. B. Ford (Chairman), University of Michigan; Professor S. Lefschetz, University of Kansas; Professor W. R. Longley, Yale University, and Professor R. E. Root, U. S. Naval Academy. This committee is already considering plans for its operation and will make a public statement through the columns of the MONTHLY within a short time. In general, it may be said that the Association proposes to take a lively interest in the development of mathematical libraries in the colleges and will wield its influence as far as possible toward this end.

At about the time when this issue of the MONTHLY reaches the members of the Association, the nomination ballots will be received from the Secretary, and the first opportunity will be presented for selecting candidates for the officers of the Association by open primaries. This provision of the Constitution is in the nature of an experiment and its successful operation depends entirely upon the thoughtful interest shown by each member. The nominations will close on November 20 and hence prompt action is needed in returning the ballots.

Four sections of the Association have thus far been organized, namely, in Kansas, Ohio, Missouri, and Iowa. Aside from the organization meetings, the Ohio and Kansas sections have each held meetings in which successful programs were carried out. The Missouri Section will conduct its first program on Saturday, November 16, 1916, at St. Louis, Missouri, and the Iowa Section is likewise planning to hold a meeting soon. It seems clear that the gatherings in which the great body of the membership can take part must be in the smaller geographical units rather than in the national meetings. Gratifying as was the attendance at the Cambridge meeting, and gratifying as the still larger attendance at New York is almost certain to be, nevertheless, these national meetings can probably never include more than one tenth to one fifth of the membership. Hence, whatever benefit is to come from attendance upon meetings and mingling with other members must, for the large majority, be found in the sectional meetings.

It should be possible for almost any member to reach some central meeting place within his own state at least once each year. Moreover, the opportunities afforded in the smaller groups for more extended discussions, especially upon those questions which involve local conditions as, for instance, legislation by states on educational matters, should be such as to attract all members of a section and to command their active participation in shaping the development of mathematical interests in their communities.

It is, therefore, confidently expected that, as time goes on, the Association will develop this phase of its activities very strongly and that numerous other sections will be organized in the near future. Already it is reported that steps

are being taken in this direction in at least three other centers, and doubtless the matter is under informal consideration in still other parts of the country. There is no need for haste, but there is need for most careful deliberation in order that any action that is taken may be permanent and effective. A watch-word that may well be considered as indicative of the spirit of this Association is "active participation on the part of all members." It is a fact of more than ordinary significance that already well toward one hundred persons are directly engaged in active service either as members of the Council, as members of the Editorial Board, or as members of standing or special committees. Indeed, if we count the officers and committees of sections already formed and in process of formation, and the contributors to the programs of meetings and to the various departments of the MONTHLY, the number of active workers must be considerably over one hundred. While this may be considered a most favorable beginning for the first year, nevertheless, the ultimate goal should be to enlist at least one thousand active participants in the affairs of the Association. This can be done as soon as the sections are numerous enough to enable every member to attend at least one meeting a year, and as soon as the opportunities for cooperation in connection with the publications of the Association and in connection with the investigations of its standing committees are fully realized. The great work of this Association will doubtless be done through its sectional meetings, where personal contact and individual responsibility will have full play, through its standing committees which in time are sure to influence every phase of mathematical interests in the country, and through its official publications which should become the clearing-house for all these activities and the medium of intercommunication which will keep every member, however remote from the natural geographical centers, in close touch with all that is taking place in the mathematical world.

One most effective service which any member may render at once is to make known to those who are as yet out of touch with the MATHEMATICAL ASSOCIATION OF AMERICA the important events of the past ten months and to see to it that the Secretary has the name and address of any such persons. The Secretary until further notice will be at *5465 Greenwood Avenue, Chicago, Illinois.*

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